STRUCTURAL RESONANCE

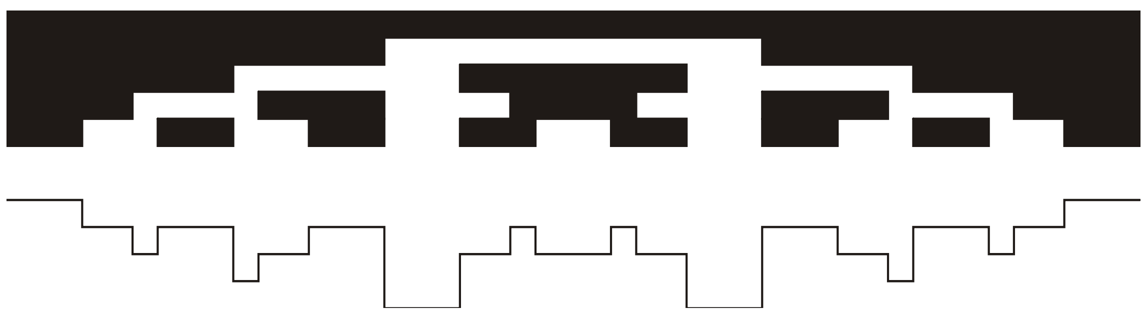
THE MATHEMATICS OF MUSIC
VIBRATION AND RESONANCE

THE MUSIC OF PHYSICS

PREFACE

By

Thomas Wagner



I consider it quite possible that physics cannot be based on the field concept, i.e., on continuous structures. In that case, nothing remains of my entire castle in the air, gravitation theory included, [and of] the rest of modern physics.

Albert Einstein
in a 1954 letter to Michele Besso

continuum: A continuous extent, succession, or whole, no part of which can be distinguished from neighboring parts except by arbitrary division.

PREFACE

The lights go down. A violinist stands onstage waiting for the audience to quiet down. The audience noise quickly abates and becomes quiet. Quiet? Perhaps only an artist onstage actually knows how noisy a completely ‘quiet’ audience really is. He draws a stand of horsehairs across a tightly stretched string. Aided by a bit or rosin the friction of the horsehair across the string creates a structure that is truly magical. This string, whose weight would be expressed in milligrams, is excited by less energy that it takes to lift a coffee cup. Even then, less than two per cent of the energy transmitted by the bow actually sees its way into the structure that is the sound. A few physically exotic things happen and the result 'fills' the hall.

On the other hand to feed a similar sound to the hall electronically a quite powerful amplification system is required. The energy the violinist expends is a fraction of a watt whereas the energy required by the amplification system will be at least a couple of hundred watts.

Why should this be? It is, or attempts to be, the same sound. The acoustical engineer points to the waveform on an oscilloscope and declares that his system creates a waveform so similar that it is impossible to distinguish it from the one created by the violin. What he says is true but still it takes hundreds of times as much energy to produce a sound electronically than it does acoustically and even then it is not a true reproduction. Something is wrong, not with the equipment but with the basic concepts we have of sound itself. These incorrect assumptions extend as well to vibration.

Even with this tremendous expenditure of energy the sound simply does not sound real. If these procedures produced a genuine, realistic sound, the energy requirements would be trivial but the two are facets of the same problem. To solve one would be to solve the other.

The primary method of analyzing the waveform that appears on the oscilloscope (and for that matter, for analyzing any periodic function) is to break the function down into simple sinusoidal components. This is, of course, Fourier analysis. This method of harmonic analysis is very well known and has been used for many things for a very long time. However, even with splendid oscilloscopes and powerful computers the result still does not create the same structure as that created by the violin. If it did the sounds would be indistinguishable.

Fourier analysis is powerful and very useful tool but it, like most mathematical procedures, describes linear functions and this paper will prove that music is non-linear. Such harmonic analysis behaves exactly as it is defined. This is not an uncommon occurrence. There used to be a sign over the desk of Igor Sikorsky that stated, "It can be proven mathematically that the bumblebee cannot fly. Fortunately the bumblebee does not know this." There was proof around the turn of the century that 'proved' mathematically that heavier than air flight was impossible. Many such examples can be found. This does not denigrate mathematics in any way, it merely illustrates that mathematics is only one part of the puzzle.

In effect this analytic procedure (Fourier analysis) breaks a complex wave up into a series of circles as, graphically, sines and cosines are most simply represented as circles. There is an even older theory that uses circles that had stood for centuries; the Ptolemaic theory of the structure is the Solar System. The reason it lasted so long is that it cannot be challenged mathematically. As a mathematical theory it is quite sound (many mathematicians at that time felt that it was too sound). What is wrong is that the basic assumption upon which it was constructed.

An even more appropriate example is the calculation of the speed of sound by Newton and Lagrange. They both used conflicting and complex methodology but ended up with the same basic result, which was wrong. Again, the problem was not the mathematics. It remained for Laplace to figure out that the error was the assumption that the elastic motions of the air particles take place at a constant temperature (isothermal volume elasticity) or, more simply put, the temperature of the air is constant during the passage of the sound wave.

Laplace figured that since the frequency of sound is so rapid that the temperature would, in fact, change as the sound wave progressed, the so-called adiabatic law. Once again the problem was not in the mathematics.

In the dichotomy of the real and electronically reproduced violin sound we have a powerful rigorous mathematical procedure combined with extremely well researched and constructed hardware that does not create the same phenomenon that the violin does. It would seem that somewhere in the system there is an incorrect assumption - - perhaps many.

Music and vibration are both discrete functions in every way. They are not, in any way, analog in nature. And here we have the beginning of the argument. It is an argument that has been going on unchanged at least since the beginning of Greek thought, and probably much longer.

CONTINUITY

Most thinking today seems to feel that we can finally explain the great Eliatic paradoxes. Consider the one about rabbit and the turtle. The rabbit starts from a point twice the distance from the finish line as the turtle. As the turtle covers half the distance the rabbit reaches the starting point of the turtle. The turtle covers half the remaining distance or ¼ of original distance. The rabbit covers ½ the original distance. The race continues in like fashion. The result is two infinite series.

$$\boldsymbol{Rabbit:}\ \frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}\ \frac{1}{32}\ \boldsymbol{etc.}$$

$$\boldsymbol{Turtle:}\ \frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}\ \frac{1}{32}\ \frac{1}{64}\ \boldsymbol{etc.}$$

It appears that the rabbit can never catch the turtle and this is said to be paradoxical. After all, we are now told that the sum of an infinite series is **1,** or at least that is the current view. The theory of a limit is the bedrock of calculus however; if this were true then there must a penultimate number, the last fraction before **1** is reached. The answer actually lies in the way in which the problem is stated. The conditions of the statement are fulfilled if, *and only if*, both the rabbit and the turtle *stop* after they reach their fractional distances. If this happens the rabbit would indeed never actually catch the turtle. The asymptote never reaches the axis it approaches. If neither stops there exists no point that can be said to be ½ or ¼ or any point whatever. The distances are continuously changing so that no defining point can be stated until they stop. Defined as such the rabbit not only catches the turtle but also overtakes it. In either case there is no paradox.

Many think that calculus is mankind's greatest invention. There are certainly good reasons to think so. Unfortunately that distinction does not give it a meaningful connection with physical reality. Calculus 'works' by producing excellent approximations. It can even create approximations for fictitious things.

In spite of its power, and at the risk of sounding facetious, we could state that calculus is the process of doing things that can't be done to things that don't exist. Perhaps it is better to say that calculus is built upon oxymorons. Take, for instance, instantaneous speed. An instant is the temporal equivalent of a point and as such has no dimension. Since speed is a function of time and distance and distance, in the case of an instant, is zero it follows that at any given instant the speed is zero. The same problems arise in the study of curved lines and it is precisely this problem that led to the development of the calculus. Instantaneous speed became the derivative of a real value function. This is one of the most brilliant creations that man has ever developed but even such a mind as Einstein has stated often that our mathematics does not reflect physical reality. Bishop Berkeley went so far as to say that infinitesimals are the ghosts of departed quantities.

If you can do it and it seems to work – why not? The explanation that calculus is counter-intuitive is dodging the issue. If we were to define a proof for the Mean Value Theorem we would begin by setting up the condition that *if* a function *f* is continuous on the closed interval [a, b] and differentiable on the open interval (a, b)……… let's stop there. In order to be continuous a function must consist of dimensionless points. If point has dimensions the function ceases to be continuous. This is expressed by [a, b]. To create the open interval, (a, b) we remove a dimensionless point from either end of the function.

The abstract nature of much of our math goes back at least to Euclid. Euclidian geometry is, in itself, abstract. A dictionary defines a point as 'thought of as location in space but having no dimensions. It then goes on to define a circle as a curved line every point of which is equidistant from a center point. There is no rational value for π because a circle is an abstraction.

The Fourier series attempts to show us everything and everything is not what we need. This paper will attempt to prove that every aspect of music is both non-linear and digital. The mathematical approach used in solving the riddle of music is, in fact, the problem itself. Linearizing a non-linear system is the primary approach taken with much of the mathematics of scientific thinking. It is the heart that drives the calculus.

This is not a criticism of the calculus, as calculus is a powerful tool and the linear approach is all too often quite necessary. It is difficult, for instance, to imagine solving a problem of orbital mechanics without using continuous functions.

The significant scientific research of the last century has showed us that many, if not all, of the things we accept as analog in nature are, in fact, discrete. The mathematics we use to work with them was designed to define all waves as being continuous. Once the existence of the atom was established matter could no longer be thought of as being comprised of a collection of infinite points, and yet, all mathematical methods of understanding the mechanics of matter treat it as though it were (consider fluid dynamics or tensor calculus).

Consider heat and light. Plank established the fact that heat was not continuous but rather was comprised of discrete packets, the quanta. Einstein did the same with light, establishing the fact that light as well was not continuous but was also comprised of packets, the photons. Mathematical descriptions of both of these, for the most part, still treat them as fields. Those pillars of physics, the Maxwell equations, are field equations, written at a time when the common belief was that electromagnetic radiation traveled on the ether and that such fields were continuous. For much of physics such reliance of continuous functions are necessary as non-linear solutions are not common. No such necessity exists for music. Music is discrete. There is *nothing* in all of music that cannot be defined by integers.

Music *is* physics. It is its own physics. It can be said that music is a branch of physics, a basic branch. It has been said that music and mathematics have much in common when actually music *is* a mathematical system in its own right. The definition of this mathematical system is complete within itself and is digital. There are no negative numbers, there is no zero and there are no irrational numbers. The beauty of music as a mechanical system lies in the fact that it can be described by mathematics composed entirely of positive integers.

Part one of this paper fully describes that mathematical system. This paper defines both mathematically and physically the basic structure of sound. In fact, anything that makes a noise does so by creating a slightly different version of the same basic structure. A device that creates this structure will create a sound experience that will be indistinguishable from the one created by these few strands of soft horsehair across an elastic material under stress or any other sound that we might imagine.

This paper also puts forth a definition of the structure that is speech. Everyone today trying to 'speak' to machines is trying to do so with wave structures that were analyzed by an oscilloscope, an analog device. The problem is augmented by the fact that the oscilloscope gives a distorted picture of, not only speech but of the musical sounds as well. It does not offer a clue to the fact that speech and music are two separate and distinct physical functions.

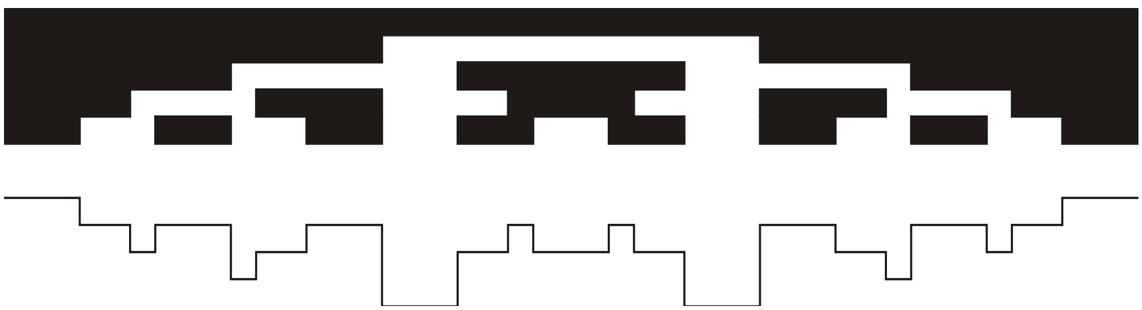
The big buzz today is ‘digital’ music. What is actually being done is the chopping up of an analog wave into small sections. In reproduction the small chunks are reconstituted into the same analog wave. This is merely an efficient way to store an analog waveform. It is in no way digital music. It is the purpose of this paper to define true digital music.

STRUCTURAL RESONANCE

THE MATHEMATICS OF MUSIC AND VIBRATION

PART ONE
THE MATHEMATICAL BASIS OF THE ENHARMONIC SYSTEM

by
Thomas Wagner



God invented the integers
man invented everything else
Leopold Kronecker

Here is the problem. There are twelve keys to the octave on the piano. Since all keyboard instruments use the same keyboard they all have the same twelve keys to the octave layout. However, in order to play a complete enharmonic system and keep everything in tune, the keyboard would require thirty-eight keys per octave. This would make the piano very hard to play and equally hard to construct.

The advantage of the keyboard is that with it you can play a bunch of notes all at the same time and play them, at least it would appear that you could, in any key. To the uninitiated a key is a precise and structured generation of frequencies. These frequencies are fixed and do not change and are numerically related by very simple ratios, ratios formed by small integers.

$$\frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{2} \quad \frac{5}{4} \quad \text{etc.}$$

The problem arises when we allow the generating tonality (or fixed, basic pitch) to change. Most of the music that has existed in the world appears to have been based on a single, unchanging generator tone. Japanese music, Chinese music, East Indian music, American Indian music and ancient Greek music, all are music systems that never change their generating frequency. Indian Ragas seem to be always in the same key. All Sitars seem to be the same size. With the exception of our own, all the music systems ever discovered have been structured on an unchanging tonal center. The idea that a note of the scale that is not the fundamental could become the fundamental of another structure apparently never arises in such cultures.

Sometimes the absolute pitch of the tonality becomes fixed in the society, that is, every piece of music a society sings or plays will be in the same 'key'. A society whose pitch is **B**♭ would remain **B**♭ for thousands of years in many cases. The need to create instruments would necessitate this even if the pitch did not remain fixed in the society by means of tonal memory.

A modern example of this is the Vietnamese language. In this language the actual pitch of the word affect the meaning. It is not surprising then that some ninety per cent of Vietnamese have absolute pitch.

There is a big advantage to an unchanging fundamental. All of the notes of the scale generated by a given fundamental are as fixed as the fundamental. In most societies there is a definite reason why the pitch of the commonly used fundamental would be constant. This will be discussed in a later section of the paper.

Music is innate in man and its mathematical structure is an absolute. We can safely assume that on any planet in the universe music would develop and it would have the same structure. There has never been a social structure discovered that exists without it. There is reason to assume that we sung as a species before we spoke. Singing is more natural, especially in tribal situations.

This innate quality is further supported by the fact that all systems have a common scale, the pentatonic or five-tone scale. A few cultures have extended this scale but the basic pentatonic scale is always present and with one rare exception it is always formed by the exact same ratios. The pentatonic scale of the Chinese, the American Indian, the Ancient Peruvian and all others are absolutely identical. There is a physical reason for this.

It is quite probable that the scales of most of the species that have existed stopped at five. We would probably all assume that most of the Paleolithic societies had pentatonic music and that is probably true. I suspect that there may have been a few who developed more complex versions of this scale. If we consider the length and stability of Paleolithic cultures we must guess that some of these societies developed music much more complex than simple pentatonic. One example is the music of India. This system is very old, was developed totally by rote and does indeed extend the pentatonic scale. We can see this in Japanese music, Bantu music and the music of the ¡Kung of the Kalahari.

Any time the scale develops to five distinct pitches we have the phenomena of the comma. The comma is a mathematical inevitability. It appears when the scale reaches at least five tones. It is the comma, more than anything else, which allows the creation of all scales and harmonic systems. It was certainly known to the ancients. The ancient Greeks defined it mathematically but there is evidence that it is quite likely that every culture that uses music (which is probably every culture) is aware of it. Incan pan flutes have pipes tuned to very small intervals. These were quite probably used for comma displacement.

It is these commas (there are several) that create the problems found in tuning keyboard instruments. The most used method of solving the comma problem in modern society is the tempered scale. Before we embark on the study of the phenomenon of the comma let us first explore the nature of tempered tuning.

THE TEMPERED SCALE

The even-tempered scale is the *diabolicus in musicum* of the modern world. Its very usefulness overshadows its dangers. Without this scale, of course, the piano would not be as effective as it is. Jazz would probably not exist and it is debatable if Wagner or Debussy would have happened, or for that matter, Scott Joplin. It has become so accepted as the foundation of musical theory that few people (including musicians) are particularly aware that such dangers exist. Even before we develop the mathematics of the enharmonic system, simple logic (and, hopefully, our ears) tells us that the tempered scale is not the most desirable solution to the problem. As with so much mathematical thinking it represents an attempt to linearize a completely non-linear system.

Music is usually defined as being an analog system. This is considered to be as intuitively obvious as motion and time. Someone once figured that it we could put a small pen on the end of one of the forks of a tuning fork and then move a sheet of paper past the pen the pen would draw a sine wave. This phenomenon is used to support the notion that the sound of a tuning fork is simple harmonic motion. This is a mistake. A swinging pendulum is simple harmonic motion but the blades of a tuning fork bend and this bending is quite different than simple harmonic motion.

The amount the fork bends is very small but the forces set in motion by the bending of hardened steel are amazing. It must also be remembered that the blades of the tuning fork do not create the sound wave directly, at least not the structure we are interested in. They create another phenomenon that actually interfaces with the airborne wave. This is a stationery wave that forms in the cylindrical part of the tuning fork.

This paper will prove that every aspect of music is both non-linear and digital. Linearizing a non-linear system is the primary approach taken with much, if not, most, scientific thinking. It is the heart of calculus. This paper establishes the fact that music requires absolutely no such linearization. Music is discrete. There is *nothing* in all of music that cannot be defined by integers.

Music *is* physics. It is it's own physics and music is a mathematical system in its own right. This mathematical system is complete within itself. There are no negative numbers, there is no zero and there are no irrational numbers. The beauty of music as a mechanical system lies in the fact that it can be described by a mathematical system composed entirely of positive integers. Part one of this paper fully describes that mathematical system

THE MUSICAL SCALE

We must define just exactly what this even-tempered scale actually is, and precisely what it attempts to replace, that being the ***enharmonic scale.*** It does not replace the ***chromatic scale,*** as is usually assumed. A just-tuned chromatic scale is quite a simple structure and is easily achieved mechanically. No solution so exotic as the tempered scale would be necessary if one chromatic scale would be sufficient for all musical requirements.

A musical scale is, by conception, a scale of function, not a scale of measurement. Measuring a functional scale with a linear ruler is self-defeating. It arose because of tuning problems inherent in keyboard instruments. It was most likely the keyboard that influenced the development of musical calligraphy, as our calligraphy is better suited to tempered tuning than to natural tunings.

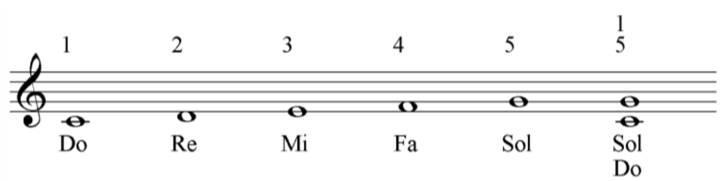
Most instruments as well as the voice are able to make the slight but discrete changes in pitch necessary to keep concerted music in tune. Even keyed instruments such as winds and brasses are able to make those shifts. The notion of an isometric scale goes back at least to ancient Greeks although it is only in recent times that it has come into common usage.

Because of the fact the he wrote a suite of preludes and fugues called The Well Tempered Clavier, the invention of the scale is often attributed to J. S. Bach. Bach did not invent this scale and in all likelihood he did not use it. A harpsichord player in Bach's time would have selected a tuning that best served the key of the piece of the Well Tempered Clavier he was playing, just as such players would likely do today. (It must be remembered that, of all the tuning systems available, tempered tuning is by far the most difficult to achieve and certainly the least satisfying.)

If we had to choose an inventor of this scale it would most likely be Andreas Werkmeister. Werkmeister theorized the scale in 1690's at just about the time that Newton and Leibniz were solidifying the calculus. This is probably not coincidence. Even the Werkmeister system used half tones that were unequal. A true isometric scale did not become accepted until late in the nineteenth century when British organ manufacturers ushered it in.

Ever since the Renaissance, scale construction has been tricky. Not only must the tones of the scale progress by intervals defined by very simple, basic ratios, but rather they must form simple, basic ratios with all other scale members as well. The intervals themselves derive their names (which themselves are functional) from their position in the diatonic scale.

Thus a fifth is defined as being the interval from **1 to 5**. In the system of **Do - Re - Mi`s** this would be from **Do** to **sol.**

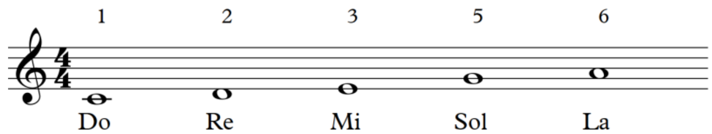


In the nomenclature of music intervals there are seven such intervals, plus their modifications (major, minor, augmented, etc.).

All sound is the product of vibration and all vibration is integral. For this reason any two frequencies sounding together form an interval. The intervals that make up music, however, are constructed from a very few basic ratios. By basic is meant ratios that are formed by two small integers. These will be completely defined in this paper. Of these few intervals, the ear demands that three of them be kept perfect. These are the fifth, the fourth and the major third.

The ear is exquisitely sensitive to the tiniest deviation from the simple, perfect ratios that create what we call music. Every culture that has ever existed has valued those individuals who, while singing or playing, keep these intervals exact. The mark of any successful musical performer on the planet is that person's ability to sing or play in tune. Placido Domingo sings in tune very well. So do Barbra Streisand and the Gatlin Brothers. In spite of the almost overriding effect the tempered scale has had upon the art in the modern era, most aspects of the art still demand proper tuning. The charm of both the string quartet and the barbershop quartet comes from the ability of the quartet members to create, between each other, the correct ratios.

The larger the number of notes in the scale the more difficult it becomes to keep all the possible intervals that can be formed ideally tuned. The most prevalent scale in the history of mankind is the pentatonic or five-tone scale.



It is not likely that a culture has ever existed that did not have this scale and it forms the basis of absolutely all other scales. In its ideal form (defined in this paper) all of the intervals formed are ideal but one.

C to **E** (do to mi) is an ideal major third; **E** to **G** (mi to sol) an ideal minor third; **D** to **G** (re to sol) an ideal perfect fourth, etc. The one ratio that is not ideal is the fourth **E** to **A**, (mi to la) which is sharper than it should be. The ideal fourth should be $4/3$ but the fourth **E** to **A** turns out to be $27/20$. This discrepancy can be shifted internally but cannot be eliminated. If **A** is lowered to a pitch that will produce, with **E**, and ideal fourth of $4/3$ we find that the fifth **D** to **A** (re to la) has now been reduced by exactly the same amount that the fourth has been raised and would now be equal to $40/27$. No matter what one does at least one of the ideal intervals will be raised or lowered, and always by the same ratio.

This small discrepancy is known in music as a comma. This comma is constant, discrete and is an integral part of any scale system. Even a scale so simple as the pentatonic cannot be created without it.

If one starts with the **C** that lies two notes from the bottom of the piano keyboard and one plays consecutive fifths above this, **C** (**C** to **G**, **G** to **D**, **D** to **A** etc.) one finds that not only do we play every note of the chromatic scale, but we end up back at **C**, now the very top note of the piano. This progression of twelve consecutive perfect fifths requires seven octaves. Mathematically octaves are the power series of **2**. Being seven octaves, if we let the initial **C** be equal to **1**, the top **C** should equal **128** (2^7). On the other hand a perfect fifth is $3/2 = 1.5$ and $1.5^{12} = 129.746337890625$. Twelve superposed fifths is higher in pitch than seven superposed octaves by the comma of Pythagoras.

$$\frac{1.5^{12}}{2^7} = \frac{129.746337890625}{128} = 1.0136432647705078125$$

This comma, **1.0136432647705078125**, is discrete and is the rational quotient of 3^{12} divided by 2^{19}.

$$\frac{3^{12}}{2^{19}} = \frac{531441}{524288} = 1.0136432647705078125$$

Structurally it is created by the interaction of two more basic commas that will be developed in the course of this dissertation.

Because of the seeming immutability of the fifth (we do after all define it as perfect) it would seem logical to assume, initially, that the chromatic scale formed in this manner is isometric. Actually, however, this Pythagorean comma (**1.0136432647705078125**) is not equally divided among the twelve fifths. Nine of the twelve fifths are indeed perfect, the other three are not.

The system is kept in tune by slight but discrete shifts in the pitch of various notes of the scale. The voice, as with most instruments, has this capability of being able to make these subtle shifts. Such instrumentalists and singers find that they are constantly altering the pitch of the various tones of the piece being played. For pianists and organists (and all other keyboard players) the experience of making music is quite a different experience. Even a keyboard player with a sense of pitch cannot affect the intonation of his instruments in any way.

Ever since the concept of the keyboard-addressed instrument arose the primary problem has been one of properly tuning it. A given generator must produce a musical tone and this generator must be pitched. If a performer wishes to play more than one note he must either change the pitch of the generator or use more generators, or both. A violin uses four generators, each of which can be modified by changing its length. Winds and brasses change the effective length of the bore as well as sounding the higher harmonics of any given length. The voice (which is truly awesome in its ability to effect the subtlest change in frequency) controls frequency by a nearly analog control of the tension of the vocal chords.

For the most part the keyboard instrument utilizes a different generator for every note. For this reason one cannot play very many notes without building a very cumbersome device. Unlike their more versatile cousins, the keyboard instruments tend to be cold, impersonal and, until the emergence of the piano, virtually expressionless. The one thing the keyboard can accomplish that makes it unique is the ability to play concerted music. The violin can play at most four notes simultaneously, and can sustain but two. Winds and brasses can play but one note at a time (apart from occasional over blowing). The keyboard, on the other hand, can sound all of its notes simultaneously. It is this capability that makes the concept of a keyed instrument so fascinating. The price the keyboard pays for this simultaneity is the loss of the ability to make those small, but extremely critical shifts in pitch.

In earlier times the tuning problem was solved by tuning the notes of the keyboard instrument, as closely as possible, to the scale tones of the piece being played. In such a case the keyboard was quite effective. As music went from the parallel tonalities of the Middle Ages to the changing tonalities of the Baroque the demands on the keyboard instrument increased.

The chromatic keyboard treats every semitone as if it were equal. We shall discover, in the course of this paper, that the true chromatic scale is actually comprised of three distinct intervals whose sequence is ordered by the fundamental tonality. The keyboard strongly influenced the development of our calligraphy, and this calligraphy, as with the keyboard, treats the chromatic scale as if it were isometric.

It is impossible to effect an acoustical tuning that allows us to keep all of the possibilities contained within a chromatic scale (for instance every possible perfect fifth) in tune. For that matter it is impossible to effect a tuning that can keep all of the possibilities of a diatonic scale in ideal ratios. We have seen earlier that even the simple pentatonic scale is unassailable. An effective way to accurately change the tuning of a keyboard instrument instantaneously has never been accomplished. No such problems exist today, however, and that provides for the basic premise of this argument.

If we let our thinking be influenced by the linearity seemingly offered by the keyboard and by our calligraphy, a hypothetical, but impractical solution suggests itself.

The admission of systems based on irrational roots into mathematical procedures, which occurred to a great degree during the latter part of the seventeenth century, profoundly affected all manner of philosophical and scientific thought. It set off serious debates that rage even today. Once such irrational numbers as the square root of two came to be thought of as being real, many things become possible. *The price for all of this, however, is the danger of losing touch with certain aspects of physical reality. Such is the case with the utilization of the even-tempered scale.*

It is important to the formation of an enharmonic scale that the series of twelve superposed fifths and the scale of seven superposed octaves be equal, ergo one of the series must be altered. The single most significant aspect of the octave is that it is an interval composed of two pitches that sound the same note. All octaves are equal to one another and are represented by the power series of two. In the mathematics of music all numbers of a binary series are the same, or more succinctly stated,

$$\mathbf{1 = 2}$$

*Even the tempered scale does not tamper with this concept***.**

A more direct way to make the series of fifths coincide with the octave series would be to shrink all of the fifths by an amount necessary to make the **129.746338** of the series of fifths equal to **128**, or $\mathbf{2^7}$. The new tempered fifth would now be equal to $\sqrt[12]{128}$ or **1.49830708**. A perfect fifth is formed by the absolute ratio $3/2 = 1.5$.

The amount of deviation between the perfect fifth and the tempered one is equal to an interval of **1.00112989.**

$$\frac{4/3}{\sqrt[12]{128}} = \frac{1.5}{1.49830707687} = 1.001129090627525$$

$$1.00112989 = \sqrt[12]{1.01364327}$$

1.00112989 is a very small interval and it is for this reason that the even-tempered scale *works* to the degree that it does.

In order to understand the mechanics of intervallic hearing let us assume a simple physical experiment. Imagine two similar sound generators, each creating exactly the same frequency. To keep things simple we will also assume that they are in phase. What we hear is a single pitch that is, quite literally, twice as loud as either generator taken separately. Now we slowly induce an analog variation in the pitch of one of the generators. We can slide the pitch up or down, it does not matter as the effects of one direction is the precise reciprocal of the other. As the two frequencies diverge things begin to change. The overall cumulative sound is decreased. At the same time the two frequencies (now separate) create a completely new structure between them.

Airborne sound is the result of cyclic variations of the air pressure. When these pressure pulses are in phase they reinforce each other. When they are not in phase interference patterns are created that cause, in turn, new cyclic variations in the ongoing sound. Each cycle of the slower frequency falls a bit more behind on every sounding. Once the middle point is reached the exact reciprocal occurs, the cycles of the slower frequency now gaining on the faster until they, once again, sound together. This new cyclic variation creates a new pitch that sounds at the same time as the frequencies producing it and becomes an integral part of the interval itself. This effect is known as differentiation and these differential results cannot be avoided.

As the tones first start to diverge this differential cycle is very long. The flow from low pressure to high pressure is too long to be perceived by the ear. As the divergence increases the frequency rate of the differential tone increases. The interval between the just tuned and the tempered fifth (**1.00112989**) lies right on the cusp between where the ear can and cannot perceive this pressure fluctuation.

Once detectable, this cyclic variation is known as a beat pattern. A beat can be defined as a detectable frequency pattern lying below that pitch which produces a sensation of pitch (about 16Hz.). If the input volume is sufficient this beat pattern can become extremely annoying and even dangerous.

The divergence between the true perfect fifth and that approximated by $\sqrt[12]{128}$ is small enough so as not to create a beat pattern that is very disturbing. Because it is the inversion of the fifth, the same is true of the fourth. Unfortunately this is not true for any other interval in the twelve-tone scale.

We can see from our experiment on the piano that the twelve notes we play when we traverse the keyboard by fifths are exactly the same as those that comprise the chromatic scale. When we shrink the series of fifths we shrink the scale of half tones as well, (which, like the series of fifths is now isometric). Each of these isometric half steps is now equal to $\sqrt[12]{2}$. While this isometric scale is exponential,

$$(\sqrt[12]{2^7}) = 1.4983070768766..... = \sqrt[12]{128}$$

the problem is made even worse by the fact that the even-tempered third, because of the nature of its formation, forms a replacement for the *Pythagorean Third* $\frac{81}{64}$ and not the preferred $\frac{5}{4}$.

The ideal major third is $1.25\ (\frac{5}{4})$ whereas the Pythagorean Third is $1.265625\ (\frac{81}{64})$. It is the value, **1.265625** that the tempered third replaces. A major third is comprised of four consecutive half steps, thus:

$$(\sqrt[12]{2^4} = 1.25992103 = \sqrt[3]{2}$$

Since the *only* major third that the ear tolerates is **1.25** the discrepancy is

$$\frac{1.25992103}{1.25} = 1.00793682$$

The beat pattern produced here is both noticeable and annoying. It beats about six times a second (it is in fact this beat pattern that piano tuners use to check the accuracy of their tuning).

Since there is no meaningful mathematical relationship between the tempered scale and the just tuned, it follows that none of the various beat patterns (and with the exception of the fifth and the fourth they are all easily discernable) will have any meaningful acoustical relationship. Because of the resulting interference patterns these beat structures diminish both the amplitude and the quality of the overall sound complex. The more complex the frequency structure the worse the problem, which is indeed cumulative.

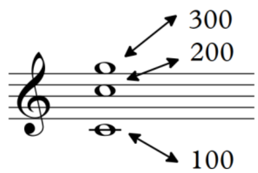
Symphony orchestras play in tune to a degree that is quite amazing. When an orchestrator *lays on sound* by adding various instrumental groupings the sound gets richer and fuller with each new addition. The reason for this is the fact that each addition is sounding the same basic structure and creates the same differential tones.

Because of the effect of the beat structures, such adding on has just the opposite effect when the added instruments are tuned to the tempered scale. Consider that one virtually never hears groups of tempered instruments. While occasionally one hears two pianos there are certainly a dearth of organ duets, or even more practical accordion duets.

The same is true for the synthesizer. The modern synthesizer is small, portable and ubiquitous and yet synthesizer ensembles are virtually non-existent. Even when a single synthesizer is used the orchestral, add-on approach actually weakens the overall sound creating a muddy, beat-laden structure that is not particularly attractive. This is what most people actually dislike about the synthesizer, and for good reason. Tuning is only part of the problem, however, as the same erroneous approach is taken with tone production, creating exactly the same problem. This will be thoroughly explored in later sections of this paper.

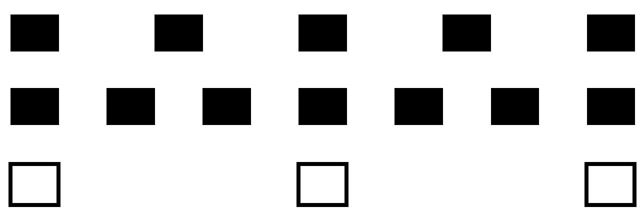
As we continue the divergence of the two frequencies the beat structure becomes fast enough to be interpreted as a tone in itself. As the divergence continues a point will be reached wherein the newly produced differential tone forms a very simple relationship with both generating tones. This is heard as a real tone and increases the amplitude of the initial interval to a surprisingly great degree. This phenomenon is known as consonance and works in the following manner.

Let us assume two frequencies tuned exactly to **200Hz**.and **300Hz**, respectively. This produces a true perfect fifth. The resultant or beat structure produced by the simultaneously sounding of these two frequencies is exactly **100Hz.** Middle **C** would be actually **264Hz** in modern tuning but allowing it to be **100Hz** makes the math easier. This produces the following musical structure:



This resultant tone is the lower octave of **200Hz**. It is this that causes the enchanting synergistic effect when true consonance is achieved. From a structural point of view the study of music is the study of the differential tone.

The top row is beating two beats in the same time period that the middle row is beating tree beats. The bottom row shows the position where the the top two rows coincide. The bottom row represents the differential.



When these patterns are actual pitches the bottom row (the differential) is very real and is heard as such. It is the only reason that we consider some intervals to be consonant and some dissonant. In the case of prime intervals this coming together is always equal to the fundamental. This 'three against two' pattern was very significant to practitioners of early music, both as the interval of a perfect fifth as well as the rhythmic figure of three beats in the same rhythmic interval as two beats. It is known as a hemiola.

When the resultant tone is not equal to the fundamental the tone formed, being a real tone, interacts with the other sounding tones (including other differentially created tones) to form still another resultant tone, etc., etc. The more obtuse the ratio of the entering tones the more complex and obtuse will be the resultant differential structure.

Mathematically the differential tone is a gift from the Gods. From the higher frequency subtract the lower frequency; the result is the differential tone. In the above example we see the simplest possible combination, **3 - 2 = 1.** Any interval made up of consecutive numbers ($\mathbf{3/2}$ $\mathbf{5/4}$ $\mathbf{16/15}$ $\mathbf{46/45}$ $\mathbf{258/257}$ etc.) will form, as its differential tone, the fundamental, **1,** and nothing else.

All of the primary intervals used in music (fifths, thirds, seconds, etc) are just such intervals. We shall refer to these intervals as **prime intervals.** A prime interval then is any interval whose differential tone is **1** (i. e., formed from consecutive integers). The reason that the scale system, which demands that all intervals formed be absolute, tolerates such *discrepancies* as appear in scales as simple as the pentatonic is that the interval of the discrepancy, the comma, is itself a prime interval.

To understand the numerical nature of this comma, consider the method of tuning an instrument utilizing four separate generators, such as the cello or violin. Because of the vertical lining up of the differential structure, differential tones sound with all other tones (differentially produced or not) to produce still more differential tones which, in turn sound again with absolutely everything, etc., etc. The more oblique the relationship between the initial frequencies the more complex will be the differential structure, the most effective points at which to stop the strings are at those points where the differential tones are the simplest, most basic ratios with the generating tones and with each other.

In the following chart we will consider the first vertical column of four separate generators, each tuned a perfect fifth apart. $\mathbf{3/2 = 1.5}$ This is the method of tuning employed by violins, violas and cellos.

Again we will let **C** = **100.**

A is higher than **D** by a perfect fifth $\quad \frac{A}{D} = \frac{337.5}{225} = \frac{3}{2} = 1.5$

as is **D** over **G,** $\quad \frac{D}{G} = \frac{225}{150} = \frac{3}{2} = 1.5$

and **G** over **C** $\quad \frac{G}{C} = \frac{150}{100} = \frac{3}{2} = 1.5$

The table shows the first six overtones of each generator.

C	**C**	**G**	**C**	**E**	**G**
100	200	300	400	**500***	600
G	**G**	**D**	**G**	**B**	**D**
150	300	450	600	750	900
D	**D**	**A**	**D**	**F♯**	**A**
225	450	675	900	1125	1350
A	**A**	**E**	**A**	**C♯**	**E**
337.5	675	**1012.5***	1350	1687. 5	**2025***

We can see the comma between the **E** of the **A** row (**1012.5**), the third overtone of **A** and the **E** of the **C** row (**500**) the fifth overtone of **C.** If we bring the **E** of the **C** row (**500**) into the octave of the **E** of the **A** row (**1012.5**) we can observe that there is a discrepancy between the two differing values of **E.** Since **1 = 2**, **500 = 1000**, and thus the discrepancy,

$$\frac{1012.5}{1000} = 1.0125 .$$

This number (**1.0125**) is the decimal value of the prime interval $\mathbf{81/80}$.

$$\frac{81}{80} = 1.0125 .$$

If we were to add a fifth generator, which would be **E,** to the complex we see that its value would be **506.25** (**337.5*1.5= 1.5) T**his conflicts with the **E** of the **C** column in which **E** is **500** producing the same discrepancy.

$$\frac{506.25}{500} = 1.0125 .$$

This is the same problem we discovered in the pentatonic scale. In a properly tuned pentatonic scale the fourth **E** to **A** is imperfect. It is sharper than a perfect fourth by the same ratio, **1.0125**

$$\frac{27/20}{4/3} = \frac{1.35}{1.33333..} = 1.0125$$

If a viola is playing on the **C** string but the controlling tonality is **E** the violist must play the **E** on the **C** string sharper than if it were controlled by a tonality in **C** and the raised pitch will be exactly **1.0125** times the **C** controlled pitch. Many things can control the pitch. In modern music the pitch is usually controlled by an aggregation of instruments with a standard pitch. A drone (such as East Indian music) often controls it and often it is controlled by the environment in which the music is played. Actual free melody (that is, with no controlling influence) while fairly uncommon allows for easy linear displacement of the comma. Linear displacement is the first method of comma displacement that evolves. This is why studies of people singing scales as being an indication of preferred tunings is irrelevant. Without an underlying pitch control a scale is apt to be most anything, yet it is these very studies that cause some physicists to insist that tuning is unimportant because people do not sing a consistent scale.

This study would be more significant if a singer were allowed to first sing in the open with no other sound. Then allow the same singer to sing the same scale in an area that supports the structure of the scale being sung (such as a good concert hall). The results will be self-explanatory.

An excellent example of linear comma displacement is the medieval organum. Singing the same melody a perfect fifth apart creates the organum. In order for the consecutive fifths to remain pure the **A** (if the governing pitches are **C** and **G**) of the lower melody is lower than the **A** of the higher melody by **1.0125**. The organum is usually sung in an area that is highly echoic and this area provides an excellent natural accompaniment. This is a big reason for the exciting sound of the organum. It is an example of true polytonality.

As mentioned previously this discrepancy can be moved around in a properly tuned complex but *cannot* be avoided. When one member of a string quartet adjusts the pitch of the note he is creating by a small amount in order better to tune with other members of the quartet, the amount of adjustment he will effect is in virtually every instance equal to exactly **1.0125.**

We call these small discrepancies (and there are several) commas. We shall discover other commas of great import, but none quite so valuable as this one, the age-old **comma of Dydimus.**

$$\frac{81}{80} = 1.0125$$

THE BASIC INTERVALS

Everything in music is generated exponentially. We have just observed the first example, the octave. The octave is so named because it is the eighth step in a scale of seven steps. More important to this discussion is the fact that the ratio of the octave is $\mathbf{2/1}$.

It is unique among intervals in that both notes of the interval are heard as being the same note. That means that structurally $\mathbf{2/1 = 1/1}$. Neither interval produces a differential tone and they are the only intervals that do not do so. *All* of the energy of **2** goes into strengthening every cycle of **1.**

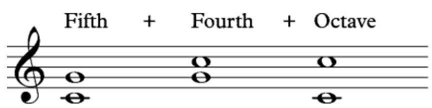
Because of this quality $\mathbf{2/1}$ must become the limit of the scale. Whatever we do, it must be done between **1** and **2,** for at **2** it all begins again.

One of the big problems that arise in understanding music comes from the belief that the octave, or $\mathbf{2/1}$ defines a *distance* and that scale steps are units of linear measurement. A scale is the product of a series of interlocking intervals which, when multiplied together equal **2.**

A musical scale is a scale of ***function;*** *it is* ***not*** *a scale of measurement.*

A scale tone is a function of where it is going and what is sounding with it.

When intervals are superposed, that is, when the upper tone of one interval becomes the lower tone of another, collectively they form a larger interval that is the product of the intervals themselves. A perfect fourth superposed with a perfect fifth will produce an octave.



$$\frac{3}{2} * \frac{4}{3} = \frac{12}{6} = \frac{2}{1} = 2.0$$

This example illustrates the basic conflict between structural and linear approaches to resonance. In a resonant structure a note **1.5** times the fundamental **(1)** marks the point *halfway* between **1** and **2.** In a linear system a point halfway between **1** and **2** is $\sqrt{2}$.

If we examine the previous example in a higher octave

$$\frac{12}{6} * \frac{16}{12} = \frac{192}{96} = \frac{2}{1}$$

We can observe the following; between **8** and **12** are three overtones **9, 10, 11**. Also between **12** and **16** are three overtones **13, 14, 15.** We shall discover that because they are inversions of each other, the fifth and the fourth are versions of the same interval. Structurally the central point of the octave is that point that lays **1.5** above the fundamental.

An isometric central point is a physical impossibility. A frequency, by definition, is both discrete and integral and there exists no integer which when multiplied by itself will equal **2.** The note in the tempered scale that is represented by $\sqrt{2}$ is the tritone. In the chromatic scale the tritone is the note most remote from the generator or fundamental.

If **1 = 2** it follows that any octave within the series is equal

$$\frac{2}{1} = \frac{16}{8} = \frac{1024}{512}$$

We may consider $2/1$ as being the prime octave. It is not until the second octave, $4/2$, that an integer appears between the terminal points of the octave.

Since **1 = 2** all versions of the power series of **2** are the same note. This structure $2/1$ has great acoustical value but little musical value. To create music we must be able to sound different notes.The first such division of the octave occurs in the second octave $4/2$ and is, of course, **3.**

2	4
	3
1	2

The ratio $3/2$ is the next interval after $2/1$ in the hierarchy of ratios, or intervals, to use the musical terminology. This interval is known in music as the perfect fifth (again named for its position in the seven-tone diatonic scale). The top half of the octave is the ratio $4/3$. This interval is third in the intervallic hierarchy and represents music's perfect fourth. An overtone cannot exist in any octave before its actual entry as an integer, that is, **3** cannot exist in the octave $2/1$. However, if the ratios are expressed as decimals a more efficient way is produced to mathematically understand music's various phenomena. Thus $3/2 = 1.5$; $4/3 = 1.3333...$ and $2/1 = 2.0$. The structure **2, 3, 4** becomes **1, 1.5, 2.** The importance of the decimal version becomes apparent if one considers the fact that the inverse of the structure, **1, 1.333333, 2** cannot be stated in integers by a simple octave shift.

The fifth and the fourth are considered to be invertible and are also considered to be inverted versions of each other.

$$\frac{2}{1.5} = 1.33333...$$

$$\frac{2}{1.33333} = 1.5$$

$$1.5 * 1.33333... = 2.$$

The inversion of an interval is accomplished by raising the bottom frequency by one octave (multiply it by **2**), or reducing the upper frequency by one octave (divide by **2**). Thus the inversion of the perfect fifth $3/2$ becomes the perfect fourth $4/3$ and the fourth $4/3$ becomes the fifth $3/2$. This is made possible by the fact that **1 = 2.**

This division, **2, 3, 4** or **1, 1.5, 2** is the first example of what must be considered to be the ideal structure in music, that is, a prime interval that is the product of at least two smaller prime intervals. In this case the interval $2/1 = 3/2 * 4/3$. All of the intervals in common practice harmony have this in common. We shall see that the entire system interlocks.

If **2** in this structure is **C** then **3** is **G.** Recalling for a moment Kronecker's words, consider the fact that any finite frequency will have an integral relationship with **C,** regardless of the value of **C.** However, if we create a perfect fourth above **1** in the octave $2/1$, it will have a value of **1.3333,** which does not have an integral relationship with **1.** What this means is that the fourth note of the scale of **C**-major (which would be **F♮**) is a note that cannot be produced in a system generated by **C.** This is a very important clue to the makeup of the enharmonic system.

THE THIRD OCTAVE

The third octave, $8/4$, presents a somewhat more interesting structure.

4	**8**
	7
3	**6**
	5
2	**4**

Now we have two new numbers, **5**, and **7**. **6** is the octave of **3** and, since **1 = 2**, is the same as **3**. **4** and **8** are equal to **2**. We have the first structure, **4, 3, 2** represented as **8, 6, 4**. Both the fifth, $\mathbf{3/2}$ and the fourth, $\mathbf{4/3}$ are again divided into intervals that are equal in that each interval increases by a number of overtones numerically equal to the fundamental. In this structure that is **1**.

Reduced to the prime octave, $\mathbf{2/1}$, (in this case derived by dividing each number of the above structure by **4**) the structure is:

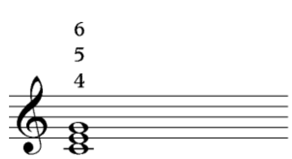
8	**2**
7	**1.75**
6	**1.5**
5	**1.25**
4	**1**

Here we see a consistency in numbers of the overtone series of **2**. Any number in the system will be a decimal ending in **5** when brought within the prime octave. The number of digits to the right of the decimal point indicate which octave the value in question first appeared, this octave being equal to the number of digits to the right of the decimal point plus **1**. Thus **1.5** enters the system as **3**, which is in the second octave. **1.25** and **1.75** enter as **5** and **7** and are both in the third octave.

The task is made easier by the fact that we already have both the pentatonic and diatonic scales and we have them both in completely accurate form. To that extent we know what we are looking for. The ratio $\mathbf{5/4}$ exists as a major third (with a value of **1.25**) thus if **C = 1** this note is **E♮**. **7** on the other hand does not appear in any of our scales. This is not unusual, as most integers of the overtone system do not become part of the scale. In fact the number of actual scale members is very small.

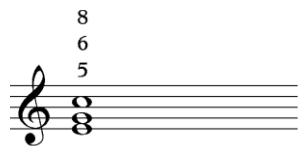
All scales are concerted in nature, that is, they are so designed that all of the notes of any scale can be sounded simultaneously. Any new entry must form simple, musical relationships with all of the other scale members. For this reason initially, **7** does not work. This will become cleared as we progress. If **C = 1**, then **7** lies very close to the note **B♭**. As we develop the chromatic scale we shall discover a better value for **B♭**.

The complex **4, 5, 6** is the prime value of the root position of the major triad.



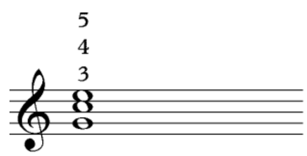
In the prime octave this is **1, 1.25, 1.5**. Since **1 = 2** the triad can also be defined as **1, 5, 3**. While **4, 5, 6** is considered to be the root position of the major triad, it is not the simplest version. The complex **4, 5, 6** is defined in music as being a chord. A chord can be defined as an interval that is the product of two or more intervals. In the triad, the fifth $\frac{6}{4}$ *or* $\frac{3}{2}$, which forms the outer two notes **C** and **G** is the product of the major third $\frac{5}{4}$, and the minor third $\frac{6}{5}$. Decimally this would be $1.25 * 1.2 = 1.5$.

A chord, just as with an interval, can be inverted, either by multiplying the bottom note by **2** or dividing the top note by **2**. This process is known in music as octave shifts. If we raise the bottom of **4, 5, 6** we get **5, 6, 8** or **E, G, C**.



This inversion of the triad is known as the six chord, so named because of the interval formed by the outermost notes, **E** and **C**. Harmonically this version is considered to be a weaker version of the triad than the root position but it remains functionally a major triad in all respects.

On the other hand if we lower the top of the triad **4, 5, 6** we get **3, 4, 5**, which, because of its position in the overtone series can be said to be the prime position of the triad.



This six-four chord so named because it is constructed intervallicly from a sixth and a fourth (**E - C** and **G - C** respectively),

This six chord is usually considered to be the weakest version of the triad even though its overtone position would indicate otherwise. This so-called weakness is actually a tonal ambiguity caused by so powerful a tone as the third overtone being in the bass of the chord. This effect, which is a keystone in the tonal structure of music, will be explained in the course of this paper.

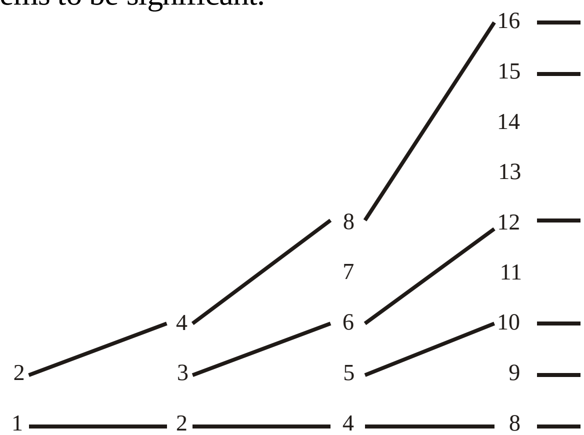
Regardless of this inversion the fundamental always remains **C**. It could never be **G** for one simple reason. If **C = 1** then **C** (**4**) in the complex **3:4:5**, lies a perfect fourth above **G** (**3**). Since the decimal value of the fourth is **1.333333**, it is physically impossible for it to appear in the overtone structure of **G**. **C** can contain **G** in its structure but **G** can never contain **C**. As far as the integral structure of the overtone series of **G** goes, **C** does not exist. It is this concept that ultimately led to the development of the diatonic scale.

The powerful and synergistic force resulting from intervals made up of simple ratios is caused by the fact that they create differential or resultant tones of equal simplicity.

THE FOURTH OCTAVE

The development of a scale must be the compound effect of what one actually hears in a sounding overtone series. Those tones we consider musical are always simple in structure and thus we are able to hear the various octaves just as the consecutive integers define them. All cultures appear to have developed at least the pentatonic scale. When scale systems become more complex, such as ours, the system is always an extension of the pentatonic. In so many cases a culture will play the entire panoply of its existence with nothing more than the pentatonic scale. This scale is so ubiquitous because it, like music itself, is innate in man. We hear this scale, to one degree or another, in every sound we hear. The reasons for this will be described later in these papers.

A pattern has developed wherein a prime interval is further divided into two prime intervals, only the lower of which seems to be significant.



The overtones of the fourth octave are thus:

8	**16**
	15
7	**14**
	13
6	**12**
	11
5	**10**
	9
4	**8**

The ratio $\mathbf{9/8}$ gives us a diatonic whole step or major second. We shall consider this to be the large major second. It produces a prime octave value of **1.125**. It is also a perfect fifth above **G** and is equal to $\mathbf{3^2}$, the power series of three being the same as tuning in fifths. $\mathbf{10/9}$ produces another major second with a prime octave value of **1.11111**. We shall consider this to be the small major second. While the large major second is the ideal in actual practice the two major seconds are interchangeable. The difference between the two is the Dydimus comma, **1.0125**.

$$\frac{1.125}{1.111111} = 1.0125$$

It is the ability to interchange these two major seconds (along with two minor thirds that we shall discover later) that permits the diatonic system to remain in perfect tune.

11, although less disruptive than **7**, does not fit into the system for the same reason as **7**. **11** in the prime octave would be **1.375**.

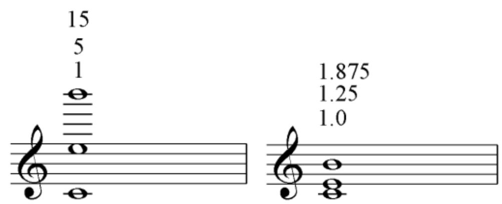
$$\frac{11}{8} = 1.375$$

However the fourth that we will use is the inversion of the fifth, which is **1.333333**.

$$\frac{2}{1.5} = 1.333333$$

As with most integers, **11** simply does not fit into the system. **13** is not used for precisely the same reason. **14** is, of course, the first octave of **7**. **15** is another story.

To start with it is a perfect fifth above **5**, ($5*3$).



Tuning in fifths is equal to the power series of **3**. **15** would be **B** in the diatonic scale of **C=1**. It produces, with **C**, the ideal minor second, **1.06666666**:

$$\frac{16}{15} = 1.0666666..$$

The first four notes of the pentatonic scale thus far are **8, 9, 10,** and **12** or, in a more structured version, **1, 9, 5, 3**. While it might appear to be a logical choice, **15** does not appear in the pentatonic scale. While **15** is the next integer to appear in the system it must be remembered that the scale in music is not a linear scale, it is exponential and the defining exponential series is the power series of three. Three of the integers of the pentatonic scale are part of the power series of three, **1**, **9**, and **3**, or $3^0, 3^2$ and 3^1. For reasons that will become clear as the structure becomes more defined the fifth note of the pentatonic scale is 3^3 or **27**. This produces an **A** of **1.6875**

$$\frac{27}{16} = 1.6875$$

This forms an ideal major second with its lower neighbor, **G** of **1.5.**

$$\frac{1.6875}{1.5} \quad 1.125$$

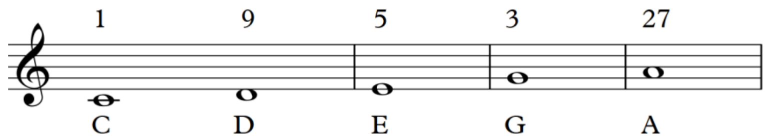
It forms a perfect fifth with **D** of **1.125**.

$$\frac{1.6875}{1.125} = 1.5$$

And, of course, it forms a major sixth with **C**, **1.6875**.

$$\frac{1.6875}{1} = 1.6875$$

This structure, **1, 9, 5, 3**, **27**, *is* the Pentatonic Scale.



A = 1.6875
G = 1.5
E = 1.25
D = 1.125
C = 1.0

This scale exists in the fifth octave, $\frac{32}{16}$ as **16, 18, 20, 24, 27**. These values, as do all other values in the enharmonic system, represent an absolute; *there is no other pentatonic scale*.

As previously observed this introduces the comma, **1.0125** into the fourth that lies between **E** and **A**.

$$\frac{1.6875}{1.25} = 1.35 \quad \frac{1.35}{1.33333} = 1.0125$$

All other intervals produced by this scale are ideal. Four of the scale members are successive generations of the power series of **3**, which is, **1, 3, 9, 27**. Let us list the scale with the power series of **3** on the vertical and **5** on the horizontal.

(81)	
27	
9	
3	
1	**5**

This is the best structural representation of the scale that we shall find. We can see that the more harmonious **5** actually replaces the next step in the power series of **3**, that is 3^4 or **81**. What that means acoustically is that the ear accepts **5** as being a more satisfying version of the major third than 3^4. The reason it does so is because the difference between the two intervals is, in itself, a prime interval, that interval being **1.0125**, the Dydimus comma. The ancients were fascinated by this ratio and in fact the major third represented by **81** is referred to as the Pythagorean third. If brought within the prime octave $\frac{2}{1}$, **5 = 1.25**, and **81 = 1.265625**, the difference between the two is **1.0125**. As we shall see this basic structure underlies, not only all of music, but all of sound.

THE RECIPROCAL NATURE OF THE DIATONIC SCALE

Two things are apparent in the pentatonic scale: a, the ratio $\frac{3}{2}$ (tuning in fifths) is clearly *the* fundamental ratio in all music and b, all of the scale members relate integrally with both each other and the fundamental, **1 (2)**. This provides an excellent definition of a scale, that is, a division of any interval, usually the octave, into a series of smaller intervals each closely related to each other and a common fundamental.

As cultures become more melody oriented (and most certainly not all do) the gaps in the pentatonic scale become more evident and steps are taken to fill in the two minor thirds that exist between the third and the fifth and between the sixth and the octave. If **C** is the fundamental these thirds lie between **E** and **G** and between **A** and C_2.

An ideal minor third is equal to **1.2.**

$$\frac{6}{5} = 1.2$$

However, the minor third that exists between **A** to **C** (between steps **6** and **8**) in the pentatonic scale produces **1.185185**...($^{32}/_{27}$)

$$\frac{2}{1.6875} = 1.185185185..$$

Once again the difference between the two is the Dydimus comma, **1.0125**.

$$\frac{^{6}/_{5}}{^{32}/_{27}} = \frac{1.2}{1.185185185} = 1.0125$$

In vertical (harmonic) structures this comma is tolerated in just two intervals, the minor third and the major second. This particular minor third ($^{C}/_{A}$) can be avoided in the pentatonic scale by simply not sounding the upper octave. As with the fourth ($^{A}/_{D}$) this can comma cannot be eliminated but only shifted within the structure itself.

If **7** were to appear in the system this is where it would be, as **A♯** or **B♭** with a prime octave value of **1.75**. One advantage in developing the mathematics of this system is that evolutionary forces have already created the pentatonic and diatonic scales in their proper form. What we must formulate are the chromatic and enharmonic scales. For this reason we already know that **7** will not work in the system and that the note we need in this spot is **B♮**. This **B** will lie a perfect fifth above **E** (**1.25**). **E** equals **5** and thus a perfect fifth above **E** would be $E * 3^1 = 15$, and would have a prime octave value of **1.875**.

$$5 * 3 = 15$$

$$\frac{15}{8} = 1.875$$

$$1.25 * 1.5 = 1.875$$

It forms a small major second with **A**.

$$\frac{B}{A} = \frac{1.875}{1.6875} = \frac{10}{9} = 1.1111111$$

This **B** forms a major third with **G**

.

$$\frac{B}{G} = \frac{1.875}{1.5} = \frac{5}{4} = 1.25$$

And finally it introduces the ideal minor second with **C**, 1.**066666.**

$$\frac{C}{B}=\frac{2}{1.875}=\frac{16}{15}=1.0666666$$

The major seventh is formed by **B** and **C**.

$$\frac{B}{C}=\frac{15}{8}=1.875$$

This major seventh introduces a new experience into the sound and structure of intervallic relationships. In common practice harmony it is considered to be a dissonance. The notion of dissonance is completely subjective; in reality there is no such thing as dissonance, merely higher numbers.

The sound of the major seventh is totally different from anything we have thus far encountered. The awesome serenity of the sound produced by those intervals we class as consonant is gone in the major seventh. The primary reason for this is that the differential tone produced by simultaneous sounding of **15** and **8**, is **7.** Bringing **7** into the system as a primary differential tone creates a complex internal beat structure that one finds either exciting or disturbing. This is entirely a matter of taste, much as with hot spices.

Extending the musical scale beyond the pentatonic appears to be an act of extreme sophistication, as it does not appear to happen often.

If we put **15** into the proper spot in the scale structure we get:

(243	
(81)	
27	
9	
3	**15**
1	**5**

15 has the same mathematical relationship with **243** as **5** has with **81**.

$$\frac{81}{1.0125}=80(5*4)$$

$$\frac{243}{1.0125}=240(15*4)$$

The ear accepts the structure as being a continuation of tuning in fifths. If this holds true the next note entering the scale should be **45**, or $15*3$. **45** in the prime octave is equal to **1.40625.**

$$\frac{45}{32}=1.40625$$

Indeed, **1.40625** does fall between the **E** of **1.25** and **G** of **1.5.** It lies an ideal major second above **E**

.

$$\frac{1.40625}{1.25}=\frac{9}{8}=1.125$$

It also lies an ideal minor second below **G**.

$$\frac{1.5}{1.40625} = \frac{16}{15} = 1.0666666$$

This presents us with a problem. The minor second below **G** would have to be **F♯** and there is no **F♯** in the diatonic scale of **C**. The interval formed between **F♯** and **C** is the infamous tritone, which produces a differential tone of **13** (**45 - 32**). What we are looking for is a note that divides this minor third between **E** and **G** in exactly the opposite manner, that is, a minor second above **E**. This would produce **F** of **1.3333333**. ($1.3333333/1.25 = 1.125$) This note, **F**, would lie an ideal major second below **G:**

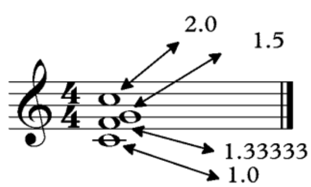
$$\frac{G}{F} = \frac{1.5}{1.3333333} = 1.125$$

This would seem to work quite well but for one thing, if **C** is the fundamental (**1**) of the scale, such a note as **F** of **1.3333333** cannot exist.

The solution is really quite simple but as yet it is apparently such a sophisticated concept that it seems to have happened only once in the evolution of the various cultures of this planet.

Let us look first at the structure itself, as defined within the prime octave:

1, 1.3333333, 1.5, 2.



If **C = 1** then the structure is **C FG C**. We can describe this structure as adjacent fourths separated by large major second:

(1 - 1.333333) 1.125 (1.333333 - 2)

$$(1.33333)^2 * 1.125 = 2\ .$$

Completing the scale means filling in the two tetrachords, **C - F**, and **G - C**. However, since the fifth is a more basic interval than the fourth we can also describe this basic structure **C FG C** as interlocking fifths.

The first fifth G/C presents no problem, since a perfect fifth is $3/2$; the bottom note of the fifth is clearly the fundamental. On the other hand the second fifth does not fit. This fifth is C/F, making **F** the fundamental. If we let **C = 1** in the structure **C FG C**, **F** cannot be defined, as it simply does not exist in the integral series of **C**, however, **C** is common to both fifths, G/C and C/F .

If, instead of **C**, we let **F = 1**, then **C** becomes the third overtone of **F** and **G** the ninth and the structure reads:

C	**F**	**G**	**C**
3	**1**	**9**	**3**

or, octave corrected:

C	**F**	**G**	**C**
6	**8**	**9**	**12**

Therefore the structure can only be **6 8 9 12**. While **8** is the fundamental, **6** appears to be the dominant note and in fact for a very long time the fifth note of the diatonic scale has been known as the Dominant. Since **C FG C** is the underlying structure of the diatonic scale it becomes obvious that the root of the scale is the fourth of the structure and that the true fundamental lies a perfect fourth above the root, or, more correctly, a perfect fifth below the root.

MUSICAL RECIPROCITY

Pythagoras himself believed that the number **3** was the beginning of all numbers; that **1** and **2** did not exist. As with the six-four chord, the basic structure, **C FG C,** actually begins with **3**. Since **1 = 2** we can rearrange the structure so as to make the **C**'s coincide.

	C	**G**
F	**C**	

	3	**9**
1	**3**	
	(1)	**(3)**

This is the key to musical reciprocity. If we consider **C** to be the dominant tone we can consider a move to **G** as being a move to the right and a move to **F** as being a move to the left. Using terms such as right and left comes from our written language and it has no meaning beyond this. The overtone series of **C** contains **G** and, in turn, **C** is contained in the overtone series of **F.** If **C = 1** then **G = 3** and **F** does not exist. If we move to the left by a factor of $3/2$**,** we then create a new note, **F,** which now becomes **1. C = 3** and **G = 9.** If we move to the right by a factor of $3/2$**. G** becomes **1** and **C** ceases to exist. Such is the reciprocal nature of music. Nothing can exist that is below the fundamental, thus a point of reciprocity can never be **1.** There are no negative numbers in the mathematics of music and there is no zero.

Let's return to the original structure, **C FG C.** Since **1 = 2** the **C**'s are equivalent and we can see that in the hierarchy of intervals an interval in root position is always followed by its inversion. The interval next in importance following the octave is the perfect fifth that, in turn, is followed by its inversion, the perfect fourth. This is verified by the sequence in the overtone series. In this series a fifth is $3/2$ and a fourth, $4/3$. In this structure we have a note, **C**, accompanied by the two tones closest to it, a fifth to the right and a fifth to the left (which is the same as a fourth to the right). This is why **C** is the principal or dominant tone even though **F** is the fundamental. This is the basic pattern upon which all western music is based.

Numerically **C FG C** is **6, 8,9, 12**. To complete the scale we must fill in the tetrachords **6** to **8,** and **9** to **12.** The shift of the fundamental from **C** to **F** creates subtle but critical changes in the structure of the scale. In most musical systems of the world the scale is generated totally by generations of an unchanging fundamental. We can think of this as starting with **1** and always moving to the right. Let us define this type of scale, such as the pentatonic, as *right-based* scales to differentiate them from those we shall define as *reciprocal* scales, such as the diatonic. We discovered in the construction of the pentatonic scale that if **C = 1** then **A = 1.6875 (27).** If we were to keep this value when we shift the fundamental to **F** it would create a major third between **F** and **A** that would be **1.265625,** the so-called Pythagorean third.

This third, created by the ratio **81:64** is higher than the ideal third of **5:4** by the Dydimus comma, **1.0125.**

$$\frac{81}{64} = 1.265625$$

$$\frac{5}{4} = 1.25$$

$$\frac{1.265625}{1.25} = 1.0125$$

If we raise **F (1.33333)** by a major third **(1.25)** we get **A** of **1.666666.** This value of **A** is lower than the right-based value of **1.6875** by **1.0125.** This is one of the more noticeable differences between right based and reciprocal scales. It is, for instance, the only significant difference between the scale of India and the western scale. It is also the difference in the value of **A** in the pentatonic scale of **C** and the diatonic scale of **C**. This difference is consistent and is *always* **1.0125.**

As previously observed, the **A** of **1.6875** lies below the top **C** by **1.185185**, the ideal minor third **6:5 (1.2)** reduced by the comma. **1.0125.**

$$\frac{2}{1.6875} = 1.1851851851$$

$$\frac{6}{5} = 1.2$$

$$\frac{1.2}{1.1851851851} = 1.0125$$

The new value of **A (1.6666)** lies below **C** by the interval **1.2**, the ideal minor third. Since this new minor third is created by a move to the left it does not exist in the series generated by **C**. In a series generated by **F**, however, this third is **5,** and, by using the first octave **(10)** we can insert it into the **C FG C** structure thus:

6, 8, 9, 10, 12

C F G A C

While there are three notes missing in the scale (**D E** and **B**), in the number system itself there are only two numbers missing, **7** and **11.** It makes us wonder about the origins of numerology and we must also wonder about just how much the ancients knew about these various structures. The numbers are small and the ratios discrete. It would not have taken a complex numerical system to understand the structures of music.

Since **C** is the principal note of the scale we can be sure that the major third above it **(E)** must be $5/4$**.** If **C = 6** this produces an **E** of **7.5 (6*1.25 = 7.5)** which is the lower octave of **15.** If we raise the structure by one octave to allow for **15** we get:

12, 15, 16, 18, 20, 24

C E F G A C

Now we can see why **11** will not work. **11,** which would become **22** in the scale must form a perfect fifth with **E (15)** while in fact, it forms an interval flatter than the perfect fifth **(1.5).**

$$\frac{22}{15} = 1.466666$$

$$\frac{1.5}{1.466666} = 1.022727272727272$$

1.466666 is not an acceptable substitute for the fifth. The fifth reduced by the Dydimus comma is only acceptable linearly. It never occurs in vertical sounding. On the other hand if we raise **15** by **1.5** we get **22.5**, the lower octave of **45** (which is now **B)** and, adjusted to accommodate **45** the scale becomes:

24, 30, 32, 36, 40, 45, 48

C E F G A B C

By the same reasoning we should expect that the **D** above the principal, **C** should be the large major second $9/8 = 1.125$. $24 * 1.125 = 27$.

Thus the smallest series of integers that can represent a diatonic scale is:

24 27 30 32 36 40 45 48

C D E F G A B C

This is the diatonic scale, as it has existed at least since the time of ancient Greece, probably longer.

It is usually defined intervallicly as fractions thus:

$$\mathbf{C}\ \frac{27}{24}=\frac{9}{8}\ \mathbf{D}\ \frac{30}{27}=\frac{10}{9}\ \mathbf{E}\ \frac{32}{30}=\frac{16}{15}\ \mathbf{F}\ \frac{36}{32}=\frac{9}{8}\ \mathbf{G}\ \frac{40}{36}=\frac{10}{9}\ \mathbf{A}\ \frac{45}{40}=\frac{9}{8}\ \mathbf{B}\frac{48}{45}=\frac{16}{15}\ \mathbf{C}$$

C (1.125) D (1.1111) E (1.066666) F (1.125) G (1.1111) A (1.125) B (1.066666) C

The diatonic scale is thus composed of only three distinct intervals:

$$\frac{9}{8}=1.125$$
$$\frac{10}{9}=1.11111$$
$$\frac{16}{15}=1.066666$$

$9/8$ defines the large major second, $10/9$ defines the small major second and $16/15$ defines the ideal minor second. Since the diatonic scale is a constant these values can be represented algebraically.

x = 1.125
y = 1.1111
z = 1.066666

C x D y E z F x G y A x B z C

The basis of this scale is two adjacent tetrachords and thus the **x** between **F** and **G** is immutable. The logical grouping of the scale intervallicly is:

xyz x yxz

Notice the slight difference between the left and the right tetrachords. The intervallic sum of each is the same.

xyz = 1.33333
yxz = 1.33333

The Dydimus comma itself is the result **x** divided by **y.**

$$\frac{x}{y}=\frac{9/8}{10/9}=\frac{1.125}{1.111111}=1.0125$$

The difference in the two tetrachords is the result of a shift of the Dydimus comma. This shift establishes the key and places the comma in the only points acceptable to the ear. This is the logical method of vertically displacing the comma. The major second and the minor third are the only two intervals in which the ear will accept the vertical sounding of the comma. The Dydimus comma is the only comma acceptable to the ear in vertical structures.

In tempered tuning both **x** and **y** are equal to $(\sqrt[12]{2})^2$ or approximately **1.122460205** instead of **1.125** or **1.111111**. **z** is equal to $\sqrt[12]{2}$ itself. It is physically impossible to establish tonality with tempered tuning as tonality is governed by shifts in the various commas and *commas do not exist in tempered tuning.*

THE TRITONAL COMPLEX

Let us examine the values of the diatonic scale as they would exists if generated by **F C** and **G**:

F	C	G
C = 2	C = 2	C = 2
z	z	Z
B = 1.875	B = 1.875	B = 1.875
x	x	y
A = 1.6666	A = 1.6666	A = 1.6875
y	y	x
G = 1.5	G = 1.5	G = 1.5
x	x	x
F = 1.3333	F = 1.3333	F = 1.3333
z	z	z
E = 1.25	E = 1.25	E = 1.25
x	y	y
D = 1.1111	D = 1.125	D = 1.125
y	x	x
C = 1	C = 1	C = 1

This was arrived at by the following method. If we let **F** (**1.333**) be the principal tone then all other notes must have the proper relationship with **F**, i. e., **G** should be a large major second above **F** and indeed **1.3333** times **1.125** equals **1.5**, which is **G**. The major third above **F** (**1.333** * **1.25**) equals **1.66666**, which is the proper major sixth. All intervals work out to be the same as in the tonality of **C** but **D**. **D** lies a major sixth above **F**. The inversion of a major sixth is a minor third ($2/1.666$ =**1.2**). **D** then should lie a minor third below **F** but $1.3333/1.2$ = **1.111111** and thus the value of **D** is changed. If we use the same method with **G** we find that every interval is consistent with the tonality of **C** but **A**, which must lie a large major second above **G**. If we multiply **G** or **1.5** by **1.125** we obtain a value for **A** of **1.6875**.

In the scale of the tonality of C notice the fact that when we changed the value of **A** from the right based value of **1.6875** to the diatonic value of **1.6666** we altered the fifth A/D . Instead of **1.5**, which is the ideal, it is **1.481481** or the perfect fifth reduced by the Dydimus comma (**1.0125**). Perfect intervals are called so because they cannot be changed. For instance, if, in performing a string quartet one player approached **A** and another **D** they would never sound the fifth as **1.481481**. They would, almost instinctively, shift tonality to the right or to the left. In either case the fifth is made pure. Consider the example of the triadic concept of the diatonic scale. Here the seven tones are arranged as three adjacent triads.

The tritone lies between the **F**, the bottom of the first triad and **B**, the top of the third triad. The diatonic scale is not a simple series of seven tones. The musical scale is not a scale of measurement but a scale of function and this is a prime example. The diatonic scale is actually a tritonal system controlled by the tonality of the center.

Consider the three structures intervallicly, grouped by tetrachords:

F yxz x yxz
C xyz x yxz
G xyz x xyz

The ear tolerates the comma (**1.0125**) when it appears in the major seconds or the minor third. The perfect intervals and the major third are kept pure by this switching of the two major seconds. Since the difference between them is the Dydimus comma (x/y = **1.0125**), the comma is shifted to a point where it will not interfere. Remember that this comma is a prime overtone interval $\mathbf{81/80}$ and thus sounds no other differential tone but the fundamental. The diatonic scale presupposes the ability to control tuning and change fundamentals. This is something that, until recently, has been completely impossible to achieve on keyboard instruments.

Like so many of man's similar attempts the keyboard arose as an attempt to introduce linearity into a non-linear phenomenon. The means now exist that will permit us to keep the keyboard in perfect tune.

Let us sum up the Dydimus comma. Notice that any time an interval was changed from its ideal value it was always by the same amount, that amount being $\mathbf{81/80}$ or **1.0125**. This was the shift required to replace the fourth power of three by **5** (**80**). Meaning no disrespect to Dydimus, we will for convenience sake rename this comma, **1.0125**, the Diatonic Comma as it is this comma, and nothing else that makes the diatonic scale work. Consider again the scale in whole numbers:

24, 27, 30, 32, 36, 40, 45, 48
or
3, 27, 15, 1 9, 5, 45, 3

If we arrange these in the same manner as we did the pentatonic and its extension we get:

27	**(135)**
9	**45**
3	**15**
1	**5**

It is structurally identical to the pentatonic scale and its extensions. It can be thought of as two superposed pentatonic scales.

9	
3	**15**
1	**5**

27	
9	**45**
3	**15**

The only difference is that in the diatonic system, **3** becomes the principal tone. To illustrate this, the Japanese use this structure in toto (even **135**). The music they produce is totally different. This is because they consider both the fundamental and the principal tone to be **1**. The reciprocal nature is not present in Japanese music even though the notes are the same.

The practice of music adheres to this math completely. **1, 3, 9, 27** and **5** are all considered to produce consonance when sounded with the fundamental. **15** (the major seventh) is consider to be 'dissonant', **45** is the devil's tritone itself and was actually once banned by papal decree and **135** which, as **C♯**, would constitute an augmented octave and thus is considered to be a false relation and is simply never used. **C♯** could only be produced by shifting to the tonality of the right, making the fundamental tone **3**, and the principle tone **9**. This, in turn, makes **135** the tritone admitting it to the system as a harmonic function.

The great cornerstone of western harmony, that moves from the tonic to the subdominant, to the dominant and then to the tonic again,

C	**F**	**G**	**C**
I	**IV**	**V**	**I**

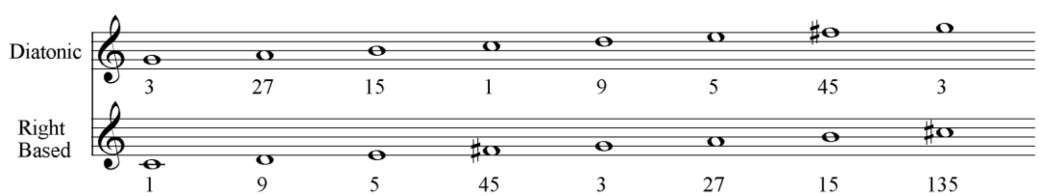
is nothing more than moves by fifths, first to the left, then to the right and then back to center. This is the function that distinguishes western music from all others and sets the stage for the magnificent enharmonic system.

THE AVOIDANCE OF THE TRITONE

The logical extension of the pentatonic and diatonic scales are, actually, the same identical structures, the only difference being which note is considered to be the principal tone. In right based scales (comprising most of the scales that have ever existed) both the fundamental and the principal tones coincide while in the diatonic the principal tone lies a perfect fifth above the fundamental.

Thus, if **C**=1, the scale represents the right based scale of **C** and the diatonic scale of **G**.

27	**(135)**
9	**45**
3	**15**
1	**5**



The Right Based scale is the scale used by the Japanese, including the **C♯**. The scale is comprised of two superposed tritones, **C** to **F♯** and **G** to **C♯**.

By filling up the 'empty' spaces in the pentatonic scale we provide a method of moving from the lower octave to a higher octave (or vice versa) by steps. The ear does not accept anything larger than the large minor second, **1.125** as a step. A minor third is already a jump or a skip.

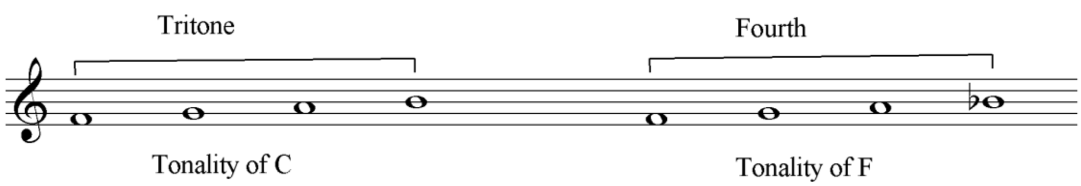
In the right based scale we notice that the fourth above **C** is indeed not perfect, nor is it affected by an interval so small as a comma. It is the only fourth in the scale that is not perfect. It is higher than a perfect fourth by a semitone. In this case, however, the semitone is not the ideal of **1.06666 (**$^{16}/_{15}$**)** but rather it is **1.0546875 (**$^{135}/_{128}$**).** Thus the interval is called the augmented fourth. As with the comma, the augmented fourth can be shifted within the scale itself but it can never be eliminated. If we change the **F♯** sharp to **F♮** l (the proper note in the diatonic scale of **C**) the fourth that is formed between **C** and **F** is perfect, but now the **F** to **B** natural forms the augmented fourth.

F♯ sharp is represent by **45** and comes into the prime octave as $\frac{45}{32}$ = **1.40625**. As with all prime octave values the value of the note in question also defines the interval itself.

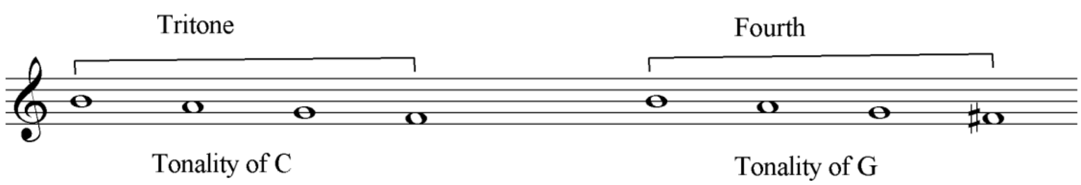
It is the only time in the scale that three major seconds occur in succession, two large and one small thus:

xyx

Because of this juxtaposition of three major seconds the interval of the augmented fourth is also known as the tritone. It was known in the Middle Ages as 'Diabolicus in musicum'. It was once banned by papal decree. It is primarily a harmonic device and in fact, common practice harmony could not exist without it. It is avoided in melodic writing in much the same manner as the diatonic comma. Now, however when the shift is made to the right or to the left, the diatonic scale of each new tonality is now used in each case. If we start with **F♮** and construct a diatonic scale, the fourth becomes **B♭** and the tritone now lies between **B♭** and **E♮**. Musically, this would be a logical move if one were ascending in the diatonic scale of **F**.



As we have seen a shift to the right changes the **F** to **F♯**. Thus if we were to descend the scale:



We know the value of the **F♯** to be **1.40625**. **B♭** would lie a perfect fourth above **F**, which in turn lies a perfect fourth above **C**. Thus in the scale of **C**, **B♭** would have a value equal to $\mathbf{1.3333^2}$. This gives **B♭** a value of **1.777777**. This is precisely the point where modern music theory goes astray. Modern theory accepts a **B♭** of **1.8,** and this value does not work as a proper value in the enharmonic system.

This value does appear, as we shall see, but not in the diatonic scale of any given tonality. If **1.8** is accepted it becomes impossible for the enharmonic system to work. It will be proven that the enharmonic scale is an absolute. One, and only one set of values can exist. The prime value of the minor seventh is **1.777777**, nothing else. **B♭** not only does not appear in the overtone series of **C** but it does not appear in **F** either. When we shift to the left and make **F** the principal tone the fundamental becomes **B♭**. Similarly, when we move the right and make **G** the principal tone the fundamental is **C**.

Another justification for a value of **1.77777** for **B♭** lies in the fact that **1.777777** is the inversion of **1.125**, the ideal major second. Just as with the perfect fifth we now have two notes created by moving the left and to the right by the same ratio. As with the fifth the value to the right or 'above' **C** lies in the overtone series of **C** and the value to the left or 'below' does not. **C** is the ninth overtone of **B♭** and **D** is the ninth overtone of **C**. This is the key to the reciprocal scale.

Consider the major second that lies between **A** and **B** in the diatonic scale of **C**. This new **B♭** divides the major second ($1.875/1.6666 = 1.125$), into two smaller intervals.

$$1.7777/1.6666 = 1.06666\ (16/15)$$

and similarly

$$1.875/1.7777 = 1.0546875\ (135/128)$$

The latter is the most remote value of the minor second and is avoided.
In music certain tones are considered to have 'tendencies'. For instance, in the scale of **F**, **B♭** is said to have a tendency to move to **A**. Once again the numbers support the practice. The ratio between **A** and **B♭** ($16/15$) is not only a more basic ratio than the one between **B♭** and **B♮** ($135/128$), but it is a prime interval. The move by the large minor second ($16/15 = 1.06666$) is always the preferred move of three possible minor seconds. **135** is **C♯** in the system-generated by **C**, and thus **B♭** to **B♮** l (**1.0546875**) would be, in music terminology, a false relation.

In the move to the right the ratios are the same but they are reversed. **F♯** (**1.40625**) lies below **G** (**1.5**) by **1.06666** ($16/15$) and above **F** by **1.0546875** ($135/128$) and again the tendency is for **F♯** to move to **G**. **F** and **F♯** are false relations.

We have two new notes and a method of creating new notes, which successfully divide the various major seconds. In this manner we will create the chromatic scale. Since the sharps and/or flats that are created are in effect substitutes for notes of the original diatonic scale, we can expect the scale to have the same characteristics as the diatonic scale, and indeed it does.

THE CHROMATIC SCALE

It takes, in effect, three distinct scales to make it possible to play a diatonic perfectly in tune. It is important to realize that while the creation of the scales on the right and on the left do add two new tones, the complete set of scales, taken together, is a functional, not a linear scale. This tritonal system creates a matrix system and not a nine-note scale, per se. We have nine notes representing a seven-note structure. If we list the tones in a linear manner (and there is a practical reason to do so) we must format them correctly, that is, as a seven-tone scale.

1, (1.111, 1.125), 1.25, 1.333, 1.5, (1.666, 1.6875), 1.875, 2

In dealing with the diatonic comma we are dealing with a very small interval. A note of **440Hz** raised by **1.0125** would be **445.5Hz**. Forgetting all physiological and psychological interpretations, the answer to the question, 'can the ear hear so small an interval?' is an unqualified yes. The hairs on the organ of Corti in the inner ear can detect minute differences in pitch, much, much smaller than those created by any of the commas we shall uncover. How the mind interprets the information is something else entirely. The point in question is that the ear always hears these differences; it cannot do otherwise.

The chromatic scale, although comprised of twelve notes, is still functionally a seven-note scale, for exactly the same reasons the simple tritonal system is. So-called chromatic notes are introduced, not to 'fill up' the spaces created by the major seconds but rather as a logical result of changing tonal centers. We have seen that if we move the left (**F**) and create a diatonic scale with **F** as the principal tone (**B♭** then is the fundamental) we introduce a new note to the system, **B♭**. Similarly the move to the right gives us **F♯**. Since these new parallel scales are diatonic scales in themselves, they must be treated as tritonal systems, each creating their own alternate tones. Even at this elementary level the system starts to grow quite complex.

We have also seen that this changing of tonal centers can be represented quite easily by creating the notes reciprocally. Thus, in this case **B♭** is the musical reciprocal of **D♮**.

$$1 * 1.125 = 1.125, \qquad \frac{2}{1.125} = 1.77777$$

We can consider this a move in either direction, right or left by the same ratio. Let us examine a much more significant interval, the major third. A major third above the fundamental is, of course, **1.25**. A major third below is **1.6** ($\frac{2}{1.25} = 1.6$). This gives us an **A♭** and, since the major third is created by the **C** above **A♭**, **A♭** is the fundamental. If the fifth to the left is second to the fifth to the right in the hierarchy of intervals, then the major third below the fundamental must follow the major third above. In decreasing importance from the generator tone the first four notes would now be:

C - G - F - E - A♭

Because of this, **A♭** is closer to the fundamental than **D** or **A**. Perhaps this can be seen more clearly in the ratios.

C = $\frac{2}{1}$

G = $\frac{3}{2}$

F = $\frac{4}{3}$

E = $\frac{5}{4}$

A♭ = $\frac{8}{5}$

The next ratio in the order is $\frac{6}{5}$ or the minor third (**1.2**). The minor third below the fundamental ($\frac{2}{1.2} = 1.6666$) already exists in the diatonic scale as **A**. **1.6666** is equal to $\frac{5}{3}$ and, although it is not a prime interval, it must be considered to be the significant version of the interval. **A** minor third above the fundamental would equal **1.2** and give us **E♭**. The fundamental of this interval would be **A♭**. Here we have two reciprocal members of the chromatic scale, neither of which resides in the overtone series of the principal tone.

The next interval in the order would be **1.125,** which we already have as **D** above and **B♭** below ($\frac{9}{8}$ and $\frac{16}{9}$). Next comes $\frac{16}{15} = 1.06666$, the ideal minor second. This, as with **A,** lies below the fundamental in its natural position giving us **B** of **1.875**. A move above the fundamental by **1.0666** gives us a **D♭** and this raises an interesting question, why do we consider **A♭**, **B♭** and **D♭** to be flats and not sharps? Flats naturally resolve to a note a chromatic half step below. It is important that this resolution be the ideal minor second. It is also important that this minor second be **1.066666**. **A♭** resolves to **G** by **1.0666** ($1.6/1.5$ = **1.06666**), **E♭** resolves to **D** by **1.06666** ($1.2/1.125$ = **1.0666**), **B♭** resolves to **A** by **1.06666** ($1.7777/1.6666$ = **1.06666**). Flats are decidedly a left based phenomenon. There are no flats contained in the overtone series of **C**, nor in any tonalities that lay to the right of **C**.

We now have eleven notes. The final note on our chromatic scale lies right in the middle, between **F** and **G**. We already have an **F♯** of **1.40625** ($45/32$). As flats go down, sharps resolve up, thus **F♯** resolves to **G** by **1.06666** ($1.5/1.40625$ = **1.06666**). This gives us twelve tones but in every other case we also produced the reciprocal of each ratio.

A move to the left of **1.40625** produces a **G♭** of **1.42222**. This is definitely a flat as it moves to **F** ($1.4222/1.3333$ = **1.06666**) by the proper minor second. This makes **F♯** quite unique, as its reciprocal is its own enharmonic substitute. Since it appears first we can assume **F♯** to take precedent. If we wish to list the chromatic scale as single values, **1.40625** will be used. Structurally, as we shall see, it is important to consider them as alternates.

Since any move to resolution that is other than **1.0666** is a false relation we can see that we still have, functionally, a seven-tone scale, just as with the tritonal complex. The chromatically scale, if linearized, must be listed so:

1, (1.06666, 1.125), (1.2,1.25), 1.333, (1.40625,1.4222), 1.5, (1.6,1.666), (1.7777, 1.875), 2

THE HEARING PROCESS

A critical question arises - can an interval so small as the Dydimus comma actually be heard? Here we must take care to differentiate between hearing and perception. If the ear is presented with a tone of **100Hz** it hears **100Hz**. If presented with **99Hz** it hears **99Hz**. If presented with an impure interval it hears an impure interval. How the mind perceives this information is much more complex. It is influenced primarily by what else is sounding with it and to a lesser degree by the experience of the performer or the listener. Tests have shown that in an interval so as basic as the octave the ear can easily detect an error of **1Hz**.

Consider the tempered major third, $(\sqrt[12]{3})^4 = 1.259921049894$. This is larger than a true major third by **1.007937**.

$$\frac{(\sqrt[12]{2})^4}{5/4} = 1.007937$$

More than any other tempered interval it is the out of tune quality of the third that gives the tempered scale its unpleasant wobbly effect. **1.007937** is a smaller interval than any of the functional commas found in the enharmonic system but it is still easily discerned as it produces a differential equally disturbing to the system. The Dydimus comma, being a prime interval ($81/80$) is much more benign to the system than any of the discrepancies that exist between the tempered and just tuned scales in that it creates, differentially, only the fundamental and thus does not beat.

In an article by Frederick Saunders published by Scientific American in 1948 entitled Physics and Music we find a clear example of the difficulties that arise when those who are not professional musicians attempt to describe the nature of musical physics. Music is as rigorous and demanding a profession as exists and, like most professions, is rife with subtleties and intricacies that the layman seldom understands and of which he is often unaware. Such is the case of Mr. Saunders offering.

Early in the article Mr. Saunders defines the frequency of middle **C** as being "about 260". That is exactly like saying that the freezing point of water is "about 30ºF". If one were to write about the nature of ice, indeed the physics of ice, one would most certainly state that the freezing point of water is 32Fº, *exactly*. Then the argument makes sense. The same is true with music.

The absolute pitch of middle **C**, if we tune to **A** of **440Hz**, could have any of four different pitches, depending upon the generating tonality. The middle **C** in a tempered scale is irrational thus we cannot express it as an integer. It would have a value close to **261.6255Hz**.

In a **C**-major scale, if **C** is the root, the value of middle **C** will be exactly **264Hz** (The reason for this value will be explained in the course of this paper). On the other hand when a violist tunes his instrument he will first tune the highest pitched string to **A** of **440Hz**. He will then tune the next lower string to a **D** of **293.3333Hz** then the next lower string to a **G** of **195.5555Hz** and finally the lowest string to a **C** of **130.3794Hz**.

This is the lower octave of a middle **C** of **260.740740740Hz**, which is **264** divided by the Dydimus comma.

$$\frac{264}{1.0125} = 260.740740740$$

Nothing is approximate in music. The error induced when we create the gamut by successive tuning of perfect fifths is very often described as being ‘about’ 2% higher than the required 2^7 (**128**) when the actual difference is **129.746337890625,** *exactly* **129.746337890625**, no more, no less. The fact that these approximations are used at all comes from the type of mathematics that is currently used to analyze and describe music.

This could be easily overlooked but for the fact that Mr. Saunders continues. He expresses the common argument that the various keys cannot have ‘character’ because of the nature of tempered tuning. However the notion of **G** and **D** being bright keys came from string music because these keys contain so many open strings. The thought that **D** was a good ‘military key’ came from the fact that military trumpets were often tuned to **D**. Also ‘just’ tuning often favored certain keys making the more remote keys have decidedly different qualities. Mr. Saunders was, as with most, thinking in terms of the tempered scale.

Mr. Saunders conveys the idea that it is the motion of the air that creates the sound. This paper will show that this concept is wrong. He states that in the clarinet and the oboe it is the lowest tone only whose sound issues from the end of the instrument. If this were true then why would a player use a mute (and both oboe and clarinet players do) if it only affected the lowest tone? Also why would the clarinet have a bell if it only affected the fundamental? This is actually a good example that shows that the movement of the air does not create the sound wave.

In his defense of tempered tuning he states that no one seems to mind when a piano and a violin play together, even though their tunings are different. Mr. Saunders should have talked to a violinist before he made this statement. All competent violinists (and most trained singers) are painfully aware of the conflicts that arise when the piano is accompanying. String quartets are universally bothered by the addition of a piano. Mr. Saunders takes the condescending view that is so often taken by ‘scientists’ when they are sure they are right and the uniformed cannot possibly have a significant argument. In all of these arguments Mr. Saunders is wrong, completely wrong.

In another statement he states that no one really hears the differences between tempered tuning and natural tuning. This is the position that is often assumed in contemporary thinking. We grow up hearing this scale and thus we cannot tell the difference is their position, or so it is thought. If it were possible for an orchestra or a string quartet to play with absolute tempered tuning we would soon find out how well the 'average' person really hears.

This is not meant to be an attack on Mr. Saunders as his place in acoustical thinking is well established. I use his article because he is considered to be a major authority,

Studies have shown that when a so-called pure tone (a sine wave) is offered to the ear the ear actually hears four harmonics. Mr. Saunders supposes that it is defect in the ear that causes these 'false' harmonics. Since one cannot hear a sine wave unless it is fed to the airborne carrier via a speaker it might be better assumed that it is the speaker that adds the 'extra' harmonics.

This, plus the Wallace Sabine's notions of hall acoustics will be taken up in a later paper.

THE TRITONAL CHROMATIC SYSTEM

A♭, which lies a major third below **C**, also forms a perfect fifth with **E♭** $(1.2/.8 = 1.5)$ (**.8** is the lower octave of **1.6**) **A♭** forms a perfect fourth with **D♭** $(1.06666/.8 = 1.333333)$. It lies a minor third above **F** $(1.6/1.33333 = 1.2)$ and a small major second below **B♭** $(1.777777/1.6 = 1.11111111)$. The only problem occurs in the fifth **B♭** above **E♭**. This fifth $(1.777777/1.2 = 1.481481481)$ is the perfect fifth reduced by the diatonic comma just as was seen in the diatonic scale between **A** and **D**.

The following table shows the chromatic scale as it would exists in the three tonalities, **F – C – G**, just with the diatonic.

Tonality in F		Tonality in C		Tonality in G	
C	2.0	C	2.0	C	2.0
B	1.875	B	1.875	B	1.875
B♭	1.77777	B♭	1.77777	B♭	1.8
A	1.66666	A	1.66666	A	1.6875
A♭	1.6	A♭	1.6	A♭	1.6
G	1.5	G	1.5	G	1.5
G♭	1.42222	F♯	1.40625	F♯	1.40625
F	1.33333	F	1.33333	F	1.33333
E	1.25	E	1.25	E	1.25
E♭	1.185185	E♭	1.2	E♭	1.2
D	1.11111	D	1.125	D	1.125
D♭	1.066666	D♭	1.066666	C♯	1.0546875
C	1.0	C	1.0	C	1.0

The impure fifth between **E♭** and **B♭** is corrected by shifts to the right or left, just as with the diatonic. While changes in the diatonic structure produce only pitch variances of the same note, the chromatic structure produces enharmonic changes wherein sharps become flats and vice versa. The **G♭** in the **F** column becomes **F♯** in the **C** column while the **D♭** of the **C** column becomes **C♯** of the **G** column.

This establishes the rather interesting fact that sharps are lower in pitch than flats. A musical scale is a scale of function. Sharps and flats exist because of where they are going and not necessarily where they are. This establishes the functional nature of a musical scale. The isometric tempered scale obscures the fact that the twelve-tone scale is diatonic in nature. The twelve-tone scale is a linear approximation and is totally irrational. *There is no twelfth root of two*.

The intervallic structure of the chromatic scale shows three intervals which we shall label **a**, **b**, and **c**.

$$a = 1.066666\ \left(\frac{16}{15}\right)$$

$$b = 1.054678\ \left(\frac{135}{128}\right)$$

$$c = 1.04166666\left(\frac{25}{24}\right)$$

Tonality in F		Tonality in C		Tonality in G	
C	**2.0**	**C**	**2.0**	**C**	**2.0**
	a		**A**		**a**
B	**1.875**	**B**	**1.875**	**B**	**1.875**
	b		**B**		**c**
B♭	**1.77777**	**B♭**	**1.77777**	**B♭**	**1.8**
	a		**A**		**a**
A	**1.66666**	**A**	**1.66666**	**A**	**1.6875**
	c		**C**		**b**
A♭	**1.6**	**A♭**	**1.6**	**A♭**	**1.6**
	a		**A**		**a**
G	**1.5**	**G**	**1.5**	**G**	**1.5**
	b		**A**		**a**
G♭	**1.42222**	**F♯**	**1.40625**	**F♯**	**1.40625**
	a		**B**		**b**
F	**1.33333**	**F**	**1.33333**	**F**	**1.33333**
	a		**A**		**a**
E	**1.25**	**E**	**1.25**	**E**	**1.25**
	b		**C**		**c**
E♭	**1.185185**	**E♭**	**1.2**	**E♭**	**1.2**
	a		**A**		**a**
D	**1.11111**	**D**	**1.125**	**D**	**1.125**
	c		**B**		**a**
D♭	**1.066666**	**D♭**	**1.066666**	**C♯**	**1.0546875**
	a		**A**		**b**
C	**1.0**	**C**	**1.0**	**C**	**1.0**

1.041666 ($^{25}/_{24}$) is the third form of the minor second.

If we group these new intervals as tetrachords as with the **xyz**'s (**xyz x yxz**) we get the following.

(ab) (ac) (a) (ba) (ac) (ab) (a)
x y z x y x z

The scale begins to approach complete symmetry.

ababc ba acaba

Grouping as with the triumvirate **xyz**'s shows.

F acaba ab acaba
C abaca ba acaba
G baaca ba abaca

In the middle structure, **C**, the tetrachords (which consist of five representing three), are symmetrical. The enharmonic scale, as we shall discover, is both symmetrical and reciprocal. The following example illustrates the proper makeup of the chromatic, tritonal system.

ac	ab	a	ab	ac	ab	a
y	x	z	x	y	x	z

ab	ac	a	ba	ac	ab	a
x	y	z	x	y	x	z

ba	ac	a	ba	ab	ac	a
x	y	z	x	x	y	z

In common practice harmony, '**a**' is the only valid half step. There is only one **a** in every half step of the diatonic scale. Singing or playing a chromatic scale in the correct tuning for a given generator is a very difficult thing to accomplish, while the diatonic is relatively easy. The chromatic scale did not arise as a melodic scale. It arose to allow proper tonal modifications to the diatonic, that is, to allow the diatonic scale to change to fit a changing fundamental. It came into being purely as a functional scale. Melodic use is usually ornamental although the art became sufficiently sophisticated as to allow the scale a genuine melodic use. Consider the 'Flight of the Bumblebee'.

Our calligraphy greatly influences the way we think about music. Invariably we think of music as moving from left to right (as with text) or bottom to top. Ask a musician to 'spell' a major triad and result will be **C E G**. Ask a person to sing a scale and they will virtually always start with the lowest note, do re mi, etc. Structurally it is just as valid to progress from left to right, or top to bottom, as this is basically what we do when we invert an interval, a chord or a scale.

An inversion of the diatonic scale illustrates the reciprocal nature of scale construction.

Ascending	**(ab)**	**(ac)**	**a**	**(ba)**	**(ac)**	**(ab)**	**a**
Descending	**a**	**(ba)**	**(ca)**	**(ab)**	**a**	**(ca)**	**(ba)**

ba

abaca **acaba**

ab

Except for the middle complex, the scales are identical. In a scale of **C**, the middle complex would be **F♯** ascending and **G♭** descending. This reciprocity will become clearer as we develop the scale numerically.

The chromatic triumvirate of scales can also be represented by **xyz**'**s**.

$$a = z,\ b = \frac{x}{z},\ c = \frac{y}{z}$$

$$\begin{array}{ccccccc}
(z\ \frac{y}{z}) & (z\ \frac{x}{z}) & z & (z\ \frac{x}{z}) & (z\ \frac{y}{z}) & (z\ \frac{x}{z}) & z \\
(ac) & (ab) & a & (ab) & (ac) & (ab) & a
\end{array}$$

$$\begin{array}{ccccccc}
(z\ \frac{x}{z}) & (z\ \frac{y}{z}) & z & (\frac{x}{z}\ z) & (z\ \frac{y}{z}) & (z\ \frac{x}{z}) & z \\
(ab) & (ac) & a & (ba) & (ac) & (ab) & a
\end{array}$$

$$\begin{array}{ccccccc}
(\frac{x}{z}\ z) & (z\ \frac{y}{z}) & z & (\frac{x}{z}\ z) & (z\ \frac{x}{z}) & (z\ \frac{y}{z}) & z \\
(ba) & (ac) & a & (ba) & (ab) & (ac) & a
\end{array}$$

THE CHROMATIC COMMA

The term, comma, comes from the Greek κόηηα which means an interval. In music this has come to mean a small interval. We shall discover several such commas. Again it must be stressed that they do not measure distance. They measure rational proportions and thus, as with any interval they are ratios. The encyclopedia Britannica lists two commas, both of which we have explored, the Pythagorean and the Dydimus. This encyclopedia states that the Pythagorean comma is higher than the mathematical ideal by **24** cents and that the just tuned third is lower than the Pythagorean by **22** cents.

The cent is the basic unit of the tonometric system. More than anything else, this tonometric system demonstrates the illogic of linear measurement in the exponential system of music. The tonometric system takes the tempered semitone, equal to $\sqrt[12]{2}$, and further divides it **100** equal parts, each part being equal to one cent. It sounds impressive and very scientific but it measures absolutely nothing. Music is described in its entirety by very simple ratios. There are no exceptions. A look at the mathematical notion of the tonometric system illustrates why it cannot be effective.

$$\sqrt[100]{\sqrt[12]{2}} = 1.0005777895065548592967925757932$$

I let result run to thirty-one places to illustrate a point. The modern organ is not particularly satisfactory. The same is true for the synthesizer. Neither has, in any sense, replaced the orchestra or, for that matter any group of acoustic instruments. The reason for this unpleasantness is simple, *they are out of tune*. All arguments for tempered tuning and a measurement system based upon it are no longer valid; they have been superseded by the computer. The tonometric system is explored in greater detail in Part III of these papers.

In the tritonal chart in the preceding section we can see that as we move from the tonality of **F** to that of **C**, the **G♭** of **F** becomes the **F♯** of **C**. Similarly the **D♭** of **C** becomes **C♯** when the shift is to **G**. This is the enharmonic shift but we are not ready to develop the enharmonic scale. The comma is rational but lies very high in the series.

$$\frac{1.4222222}{1.40625} = \frac{2048}{2025} = 1.011358024691..$$

Superpositioning would seem to be the most logical and efficient method of creating the structure of music. The most significant series is, of course, the octave series, the power series of **2**. Since **1** = **2** every member of the series is equal to every other member and all are equal to the fundamental. This definition is such that we can state that any other power series can only exist to the extent that it does not conflict with the power series of **2**.

The scale so far developed defines the pentatonic scale and its logical extension, the diatonic.

(81)	(405)	
27	135	
9	**45**	
3	**15**	
1	**5**	(25)

The series rises by powers of **3** until it reaches **81**. At that point the more acoustically acceptable **80** is substituted. The second column again rises by multiples of the powers of **3** until **405** is reached. The substitution occurs again, just as with the first column and the new column begins with **25**.

3^4	$5*3^4$	
3^3	$5*3^3$	
3^2	$5*3^2$	
3^1	$5*3^1$	
3^0	$5*3^0$	5^2*3^0

We have observed the comma between $81(3^4)$ and $80(5*2^3)$ is **1.0125,** the diatonic comma. This simplifies the system, bringing into system the significant major third $(5/4 = 1.25)$. This introduces the altered fifth.

$$\frac{3/2}{81/80} = \frac{40}{27} = 1.481481481$$

If we continue the power series of **3** in the **5** column, the value above **135** is **405**. **405** has the same relationship with **25** that **81** has with **5** and again introduces the diatonic comma. **405** in the **5** column is already affected by the diatonic comma and thus it is so affected again. The ratio of the number in the power series of **3** and the adjusted number in the subsequent column is $81/5 = 16.2$. **405** then is the adjusted value of $6561(3^8)$.

$$\frac{3^8}{81/5} = \frac{6561}{16.2} = 405$$

We must continue this expansion until the power series of **3** creates a number that conflicts with the octave.

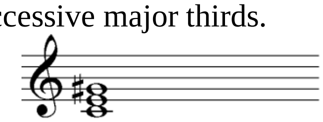
This occurs at $3^{12} = 531441$. If we expand the previous example we see a matrix pattern developing created by the power series of **3** in vertical and the power series of **5** on the horizontal.

(531441)	(524880)	(581400)	(512000)
177146			
59049			
19683			
(6561)	**(32805)**	(32400)	(32000)
2187			
729			
243			
(81)	**(405)**	**(2025)**	(2000)
27	135	675	
9	45	225	
3	15	75	
1	**5**	**25**	**125**

3^{12} **(531441)** comes very close to 2^{19} **(524288)**. The difference is equal to the Pythagorean comma.

$$\frac{3^{12}}{2^{19}} = \frac{531441}{524288} = 1.0136432647705078125$$

This is a graphical way of observing the comma caused by consecutive tuning in fifths. We see a matrix developing and this matrix will prove to be the key to virtually all acoustic phenomena. If we can superpose perfect fifths to generate scale members why not major thirds. The horizontal rows describe tuning in thirds or the power series of **5**. If **C = 1** then **5** is **E**, a major third above **C, 25** would be **G♯**, a major third above **E. 25** forms an interval known as an augmented fifth, formed by two successive major thirds.

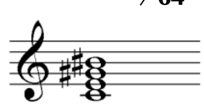


This produces a fifth augmented by the small minor second

$$25/24 = 1.0416666 .$$

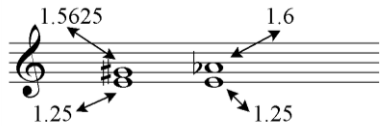
$$\frac{3}{2} * \frac{25}{24} = \frac{75}{48} = 1.5625$$

1, 5, 25 form the augmented triad. If we superpose a third major third to the structure we get **125**. This would be **B♯**. This interval would be $125/64 = 1.953125$.



Because it lies so close to the octave **C**, $2/1.953125 = 1.024$, it is forbidden in common practice harmony, which does not permit alteration of the octave. It does however provide the value of the Chromatic comma. Just as with **B♯**, **G♯** (**1.5625**) is equally close to **A♭** (**1.6**). This produces the same comma, $1.6/1.5625 = 1.024$. This is the comma that differentiates sharps and flats.

We can also see an interval between **E♮** and **A♭** which, on the keyboard, looks like a third but actually is a diminished fourth.



$$\frac{\mathbf{1.6}}{\mathbf{1.25}} = \mathbf{1.28}$$

1.28 is higher than the major third, **1.25** by the Chromatic comma, **1.024** $^{128}\!/_{125} = \mathbf{1.024}$. In the chromatic scale **1.28** would be **F♭** (**E♮** would, in this case be consider the sharp member) and the comma fulfils its function. To illustrate how everything fits, this **F♭** lies a major third below **A♭**, $^{1.6}\!/_{1.28} = \mathbf{1.25}$. The problems of a linear keyboard are exemplified here as the interval looks like a third on the keyboard but like a fourth in the calligraphy.

Not only does $\mathbf{3^{12} = 531441}$ approach the octave $\mathbf{2^{19} = 524288}$ but each successive adjustment (**32805, 2025, 125**) does so as well. **32805** comes closest, missing the octave $\mathbf{2^{15} = 32768}$ by **1.001129150390625**.

$$\frac{\mathbf{5 * 3^8}}{\mathbf{2^{15}}} = \frac{\mathbf{32805}}{\mathbf{32768}} = \mathbf{1.001129150390625}$$

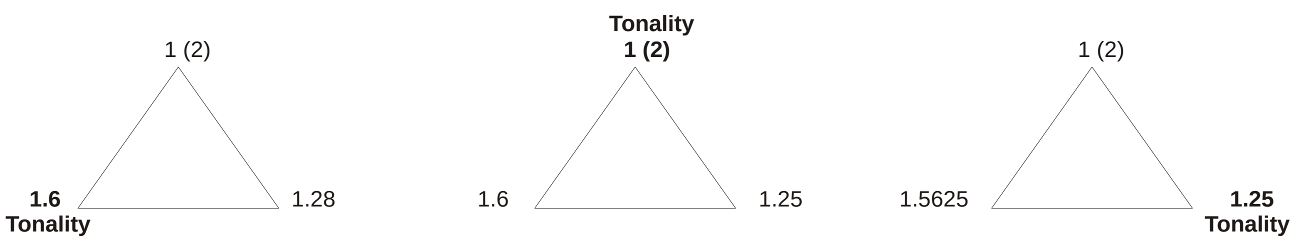
This comma, although valid and rational, does not enter into any significant musical examples. **2025** forms, with $\mathbf{2^7}$, the enharmonic comma.

$$\frac{\mathbf{2^7}}{\mathbf{25 * 3^4}} = \frac{\mathbf{2048}}{\mathbf{2025}} = \mathbf{1.011358024691358024}$$

We discovered this comma lying between **F♯** and **G♭.** Finally, **125** approaches **128** by the Chromatic comma $^{128}\!/_{125} = \mathbf{1.024}$.

This gives us all of the commas that appear in the system. It also gives us a better feeling for what actually establishes tonality.

The following example illustrates this.

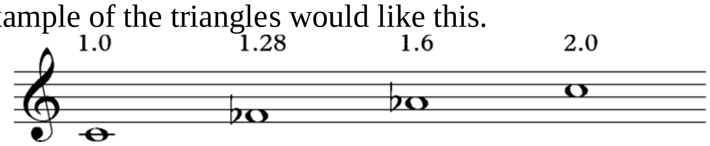


In tempered tuning these triangles would be equilateral.

The values would be consistent with each 'tonality' and would be

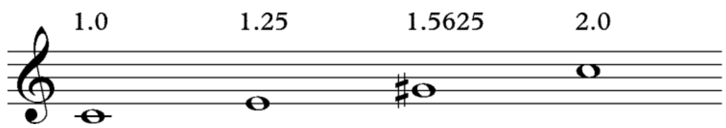
$(\sqrt[12]{2})^0$ **1**
$(\sqrt[12]{2})^4$ **1.011358024691358**
$(\sqrt[12]{2})^8$ **1.587401051968199**
$(\sqrt[12]{2})^{12}$ **2**

A calligraphic example of the triangles would like this.



Tonality in **F♭**

Tonality in **A♭**



Tonality in **C**

Augmented structures are more complex than we need to deal with in this discussion. We will use a scale construction that best illustrates the case.

F♭ xzz xy xy
A♭ xy xzz xy
C xy xy xzz

The tonality is always to the immediate right of **xzz**. The augmented triad is usually associated with the whole tone scale but this example clearly illustrates the fact that the whole tone scale is a musical impossibility. A whole tone scale would be some version of

xy xy xy

This would produce the **B♯** that we saw in the power series of **5**, in fact **xy** is **5**(**1.25**) and thus **B♯** could be written as $5^3 = 125$ and of course $125/64 = 1.953125$ which is **B♯**. We can also combine the **x's** and **y's**. An algebraic approach is used simply to manipulate the intervals. As we already have the values of **x**, **y** and **z** we do not actually solve for anything. Thus $xy\,xy\,xy = x^3 y^3 = 1.953125$. Notice also that the inversion of **B♯** (**1.953125**) produces the chromatic comma which, in the scale of **C=1** would be **D♭♭**.

$$\frac{2}{1.953125} = 1.024$$

There is no combination of **x's** and **y's** that will equal **2**. One of the **xy** structures must change to reach the octave (**2**). Since the diatonic scale is **xyz x yxz** it can be expressed as $x^3y^2z^2 = 2$. This version will prove to be very useful. Thus we find the new intervallic structure. For this we will remove the final **y** in the **xy xy xy** structure and replace it with **xz**^{2}.

$$\frac{x^3y^2z^2}{x^3y^2} = z^2$$

$xz^2 = 1.28$ and forms the diminished fourth in music terminology. $z^2 = 1.1377777777$ which, in the scale of **C=1** would be **E♭♭** and the interval described by xz^2 is the diminished third. Both the augmented triad and the whole tone scale are quite different than those defined by the tempered scale.

The diatonic comma is formed by the difference between the two major seconds $x/y = 1.25/1.111111 = 1.0125$. The chromatic comma is formed by the difference between the diminished fourth (which is more properly written as **zxz**) and the major third $zxz/xy = 1.28/1.25 = 1.024$. We shall discover that in the circle of twelve fifths the diatonic comma reduces the structure three times $(x/y)^3 = 1.0379707)$ and the chromatic comma increases it once and this in turn creates the enharmonic comma.

$$\frac{(x/y)^3}{z^2/y} = \frac{1.0379707}{1.024} = 1.011358024$$

The enharmonic comma can also be written as $z^2/x = 1.011358024$. Both the chromatic and the enharmonic commas change sharps into flats. The enharmonic shift will become clearer as we develop the complete enharmonic scale.

THE CIRCLE OF FIFTHS

We have previously observed the musical trinity, that is, a note **(C)** with a perfect fifth on either side.

F C G

While the principal tone is **C** the tonality is, of course, **F** and the smallest digital representation is **1 3 9**. We can consider either **F** or **G** as centers of their own trinities, **B♭ F C**, and **C G D**. The **C**'s are common to both and thus we have extended the initial structure, **F C G**, by a fifth in either direction.

B♭	F	C	G	E
1	3	9	27	5(81)

C remains the principal tone as we shift the fundamental to the right. Each new pair is a reciprocal pair, thus **2** divided by a value on the right produces the reciprocal value on the left. C/D = **B♭** ($2/1.125$ **= 1.777777)** . If we continue to add pairs we will generate all of the notes of the chromatic scale and eventually reach the enharmonic pair **F♯** and **G♭**.

G♭ D♭ A♭ E♭ B♭ F C G D A E B F♯

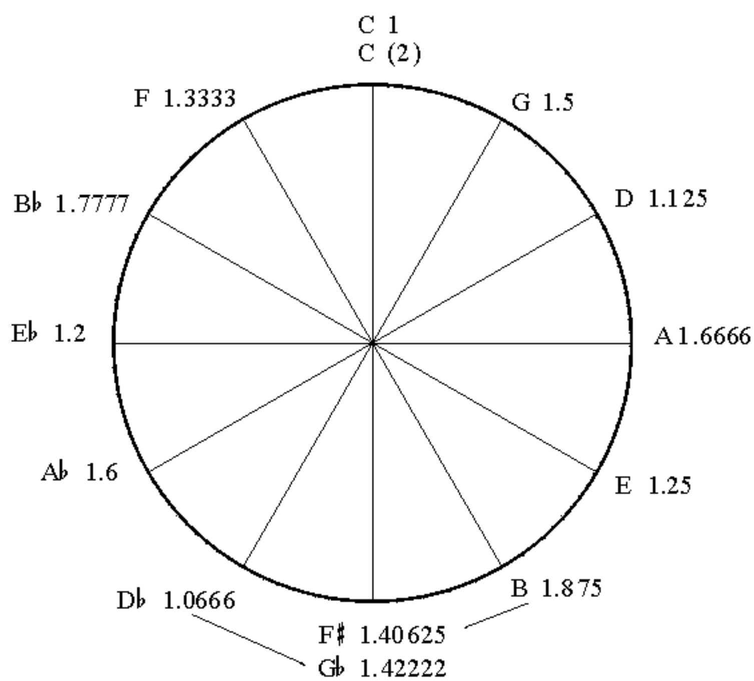
Following the logic so far set forth, **G♭** is the fundamental and **C** the principal tone. Once again we have a chromatic series formed by tuning in fifths, however, in this instance we have observed the diatonic comma and thus the octave error is between **G♭** and **F♯**. We might call this the enharmonic octave. The difference between **G♭** and **F♯** is the enharmonic comma

$$\frac{1.4222222}{1.40625} = 1.01135802$$

The inversion would produce the diminished octave.

$$\frac{2}{1.01135802} = 1.9775390625 = \frac{4050}{2048}$$

To better understand this we shall arrange the structure in the medieval style, a circle.



The value at the bottom is **F♯** if the progression is to the right and **G♭** if the progression is to the left. The values of this circle are an absolute. No other values of the circle of fifths will work in the construction of an enharmonic scale.

Nine of the fifths are perfect and three are not. If the intervals are defined as fifths the move would be to the right. If the move is to the right the **B**, five moves down, moves to **F♯**

whereas if the move is to the left the fifth move, **D♭**, is to **G♭**. If we continue to move to right **F♯** normally would move to **C♯**, and, since the comma would appear here the **C♯** would be **1.0416666**.

$$\mathbf{1.40625*1.5=1.0546875}$$

$$\mathbf{\frac{1.0546875}{1.0125}=1.041666666}$$

The actual value is the **D♭** that is the reciprocal of **B**, $\mathbf{2/1.875=1.066666}$. **F♯** to **D♭** produces a slightly augmented fifth.

$$\mathbf{\frac{2*1.066666}{1.40625}=1.5170370370}$$

This, with the perfect fifth, produces the enharmonic comma.

$$\mathbf{\frac{1.51037037}{1.5}=1.01135802469}$$

The comma can also be seen between **G♭** and **F♯.**

$$\mathbf{\frac{1.4222222222}{1.40625}=1.01135802469}$$

In the circle we observe two fifths that are reduced by the diatonic comma and one that is increased by the enharmonic comma. The diminished fifth lies quite low in the series

$$40/27=1.5/1.0125=1.481481481,$$

while the augmented fifth lies quite high :

$$1.5*1.011358024=1024/675=1.517037037037\,\mathbf{.}$$

If we start at **C** and move in either direction we will, after twelve such moves, return again to **C**. It will, of course, be in a different octave depending upon the direction chosen but, since **1=2**, it does not alter the argument.

Counting both **F♯** and **G♭**, there are six notes to right and six to the left. The values represent the true chromatic scale and the structure is both reciprocal and symmetrical. **2**, divided by any value on the right will produce its positional counterpart on the left. Lines in the diagram connect these reciprocal values.

$$\mathbf{\frac{2}{1.125}=1.777777\quad\frac{2}{1.777777}=1.125}$$

$$\mathbf{\frac{2}{1.25}=1.6\quad\frac{2}{1.6}=1.25}$$

Etc.

We can consider the structure to be a series of twelve fifths from **G♭** to **F♯**, or six fifths in either direction from **C**. The concept is not unlike that of signed and unsigned numbers.

C would be the principal tone but **G♭** would be the fundamental.

Let us stretch the circle back into a straight line and add the power series of three while observing all of the proper commas.

G♭	D♭	A♭	E♭	⊗	B♭	F	C	G	D	⊗	A	E	B	F♯
1	3	9	27		5	15	45	135	405		75	225	675	2025

The diatonic commas are marked with ⊗. This comma must be introduced after **B♭** and before **D** to maintain symmetry. In actual practice the commas would be avoided by shifts in the various tonal trinities.

45 is the tritone $\mathbf{(45/32 = 1.40625)}$. **2025** is $\mathbf{45^2}$ and thus **2025** and **45** form another tritone.

The structure that allows the circle of fifths is formed by superposed tritones. It misses the octave by the enharmonic comma.

$$\frac{\mathbf{2}}{\mathbf{1.40625^2}} = \mathbf{1.01135802469}$$

The numbers themselves form an extension of the now familiar matrix pattern. For **C=45** the structure is.

	405	2025
27	135	675
9	45	225
3	15	75
1	5	

	D	F♯
E♭	G	B
A♭	C	E
D♭	F	A
G♭	B♭	

If we express these values in the octave $\mathbf{2/1}$ we can observe the following decimal version.

	1.125	1.40625
1.2	1.5	1.875
1.6	1.0	1.25
1.066666	1.333333	1.666666
1.422222	1.777777	

This forms a very practical method of listing the values of the circle of fifths. We can now dispense with the circle and list right and left members in columns.

C	**1.0**	**1.0**	**C**
F	**1.333333**	**1.5**	**G**
B♭	**1.777777**	**1.125**	**D**
⊗	⊗	⊗	⊗
E♭	**1.2**	**1.666666**	**A**
A♭	**1.6**	**1.25**	**E**
D♭	**1.066666**	**1.875**	**B**
G♭	**1.422222**	**1.40625**	**F♯**

If we multiply each value of this structure by the fifth octave of **45** $(45 * 2^5 = 1440)$ we achieve the following.

1440	**1440**
1920	**2160**
2560	**1620**
⊗	⊗
1728	**2400**
2304	**1800**
1536	**2700**
2048	**2025**

	1620	**2025**
1728	**2160**	**2700**
2304	**1440**	**1800**
1536	**1920**	**2400**
2048	**2560**	

Since the chromatic scale is, as with the diatonic, a tritonal system we must allow for this in the matrix. This is accomplished simply by letting the principal tone of the center tonality be equal to **135** instead of **45**. This extends the matrix vertically from 3^4 to 3^6.

	3645	18225
243	**1215**	**6075**
81	**405**	**2025**
27	**135**	**675**
9	**45**	**225**
3	**15**	75
1	5	

THE ENHARMONIC SCALE

Webster's New Collegiate Dictionary defines 'enharmonic' thus: *of, relating to, or being notes that are written differently but sound the same*. This definition is, of course, wrong. Only in a tempered system, which is hypothetical, would **A♭** and **G♯** be the same. That such a definition should appear in this dictionary supports the idea that the tempered scale has so invaded musical thinking that we have to a significant degree lost sight of what music actually is.

In a tuned system **A♭** and **G♯** do not sound the same. They differ by the interval of **1.024**. This chromatic comma is a functional comma is necessary to keep the system pure. It is only in this manner that music can posses the synergistic quality that makes music so magical.

In the chromatic trinity we see four notes wherein two different values represent the same note. **D** (**1.125, 1.111111**), **E♭** (**1.185185, 1.2**), **A** (**1.666666, 1.6875**) and **B♭** (**1.777777, 1.8**). These are true enharmonic changes and in fact, define the term.

The chromatic scale is made up of thirteen notes. While **F♯** and **G♭** are never sounded together they must both be included in the system. To allow for the corrections that are necessary in vertical harmony we developed the tritonal system, producing a scale of eighteen notes. By expanding the trinity we developed the circle of fifths consisting of thirteen tonalities. We must now develop the chromatic scale for each of these tonalities. Let us return one last time to the circle for our example. For simplicity we will define the circle as columns.

This time we will rotate the circle one tonality to the left and one to the right.

Rotation left *Rotation right*

F		C		G	
1.333333	**1.333333**	**1.0**	**1.0**	**1.5**	**1.5**
1.777777	**1.0**	**1.333333**	**1.5**	**1.0**	**1.125**
1.185185	**1.5**	**1.777777**	**1.125**	**1.333333**	**1.6875**
⊗	⊗	⊗	⊗	⊗	⊗
1.6	**1.111111**	**1.2**	**1.666666**	**1.8**	**1.25**
1.066666	**1.666666**	**1.6**	**1.25**	**1.2**	**1.875**
1.422222	**1.25**	**1.066666**	**1.875**	**1.6**	**1.40625**
1.8984375	**1.875**	**1.422222**	**1.40625**	**1.066666**	**1.0546875**
⊗	⊗	⊗	⊗	⊗	⊗

If we rotate the circles, keeping the commas constant, we bring the new tonality to the top of the circle. As the various tones pass the commas their values change and the result is the chromatic scale of the initial top value. If we continue the rotate the circles until both **F♯** and **G♭** are brought to the top we will create the complete enharmonic scale.

We progress in these structures by moving the desired tonality to the top of the wheel. Thus when we move to the right the structure moves to the left and vice versa.

It is important to remember, when speaking of music, that the term 'motion' is an abstraction. Music is a succession of various pitches, nothing more, nothing less.

The complete version of the structure is necessary, as the structure is capable of moving in either direction. As with much of music, this structure is defined by where it is going.

The following charts illustrate the complete progression, first to the right and then to the left. Each tonality is shown in its complete version. This produces the complete structure for every tonality.

PROGRESSION TO THE RIGHT

C				⊗	G			
C	1.0	1.0	C		G	1.5	1.5	G
F	1.333333	1.5	G		C	1.0	1.125	D
B♭	1.777777	1.125	D		F	1.333333	1.6875	A
⊗	⊗	⊗	⊗		⊗	⊗	⊗	⊗
E♭	1.2	1.666666	A		B♭	1.8	1.25	E
A♭	1.6	1.25	E		E♭	1.2	1.875	B
D♭	1.066666	1.875	B		A♭	1.6	1.40625	F♯
G♭	1.422222	1.40625	F♯		D♭	1.066666	1.054687	C♯
D				⊗	A			
D	1.125	1.125	D		A	1.666666	1.666666	A
G	1.5	1.6875	A		D	1.111111	1.25	E
C	1.0	1.265625	E		G	1.481481	1.875	B
⊗	⊗	⊗	⊗		⊗	⊗	⊗	⊗
F	1.35	1.875	B		C	1.0	1.388888	F♯
B♭	1.8	1.40625	F♯		F	1.333333	1.041666	C♯
E♭	1.2	1.054687	C♯		B♭	1.777777	1.5625	G♯
A♭	1.6	1.582031	G♯		E♭	1.185185	1.171875	D♯
E				⊗	B			
E	1.25	1.25	E		B	1.875	1.875	B
A	1.666666	1.875	B		E	1.25	1.40625	F♯
D	1.111111	1.40625	F♯		A	1.666666	1.054687	C♯
⊗	⊗	⊗	⊗		⊗	⊗	⊗	⊗
G	1.5	1.041666	C♯		D	1.125	1,5625	G♯
C	1.0	1.5625	G♯		G	1.5	1.171875	D♯
F	1.333333	1.171875	D♯		C	1.0	1.757812	A♯
B♭	1.777777	1.757812	A♯		F	1.333333	1.318393	E♯
F♯				⊗	G♭			
F♯	1.40625	1.40625	F♯		G♭	1.422222	1.422222	G♭
B	1.875	1.054687	C♯		C♭	1.896296	1.066666	D♭
E	1.25	1.528031	G♯		F♭	1.264197	1.6	A♭
⊗	⊗	⊗	⊗		⊗	⊗	⊗	⊗
A	1.6875	1.171875	D♯		B♭♭	1.706666	1.185185	E♭
D	1.125	1.757812	A♯		E♭♭	1.137777	1.777777	B♭
G	1.5	1.318393	E♯		A♭♭	1.517037	1.333333	F
C	1.0	(1.0) (1.9775)	C (B♯)		C (D♭♭)	1.0 1.011358	1.0	C

PROGRESSION TO THE LEFT

F				⊗	C			
F	1.333333	1.333333	F		C	1.0	1.0	C
B♭	1.777777	1.0	C		F	1.333333	1.5	G
E♭	1.185185	1.5	G		B♭	1.777777	1.125	D
⊗	⊗	⊗	⊗		⊗	⊗	⊗	⊗
A♭	1.6	1.111111	D		E♭	1.2	1.666666	A
D♭	1.066666	1.666666	A		A♭	1.6	1.25	E
G♭	1.422222	1.25	E		D♭	1.066666	1.875	B
C♭	1.896296	1.875	B		G♭	1.422222	1.40625	F♯
E♭				⊗	B♭			
E♭	1.2	1.2	E♭		B♭	1.777777	1.777777	B♭
A♭	1.6	1.8	B♭		E♭	1.185185	1.333333	F
D♭	1.066666	1.35	F		A♭	1.580246	1.0	C
⊗	⊗	⊗	⊗		⊗	⊗	⊗	⊗
G♭	1.44	1.0	C					
C♭	1.92	1.5	G		G♭	1.422222	1.111111	D
F♭	1.28	1.125	D		C♭	1.896296	1.666666	A
B♭♭	1.706666	1.6875	A		F♭	1.264197	1.25	E
D♭				⊗	A♭			
D♭	1.066666	1.066666	D♭		A♭	1.6	1.6	A♭
G♭	1.422222	1.6	A♭		D♭	1.066666	1.2	E♭
C♭	1.896296	1.2	E♭		G♭	1.422222	1.8	B♭
⊗	⊗	⊗	⊗		⊗	⊗	⊗	⊗
F♭	1.28	1.777777	B♭		C♭	1.92	1.333333	F
B♭♭	1.706666	1.333333	F		F♭	1.28	1.0	C
E♭♭	1.377777	1.0	C		B♭♭	1.706666	1.5	G
A♭♭	1.517037	1.5	G		E♭♭	1.137777	1.125	D
F♯				⊗	G♭			
F♯	1.40625	1.40625	F♯		G♭	1.422222	1.422222	G♭
B	1.875	1.054687	C♯		C♭	1.896296	1.066666	D♭
E	1.25	1.528031	G♯		F♭	1.264197	1.6	A♭
⊗	⊗	⊗	⊗		⊗	⊗	⊗	⊗
A	1.6875	1.171875	D♯		B♭♭	1.706666	1.185185	E♭
D	1.125	1.757812	A♯		E♭♭	1.137777	1.777777	B♭
G	1.5	1.318393	E♯		A♭♭	1.517037	1.333333	F
C	1.0	1.0(1.97)	C (B♯)		C(D♭♭)	1.0(1.01)	1.0	C

No matter in which direction we move, right or left, the principal tone (**1.0**) is only three steps from crossing the comma. We must be careful to differentiate between the principal tone and the fundamental. The fundamental generates the tones of the structure while the principal tone defines the tonality. The fundamental of a diatonic scale in the tonality of **C** is **F**. The fundamental of a chromatic scale in the tonality of **C** is **G♭**. The tonality is established by holding the octave of the tonality true. Thus, if the tonality is **C=1.0** then the octave $\mathbf{2.0/1.0}$ cannot be altered. To do so would change the tonality.

The solution is simple. We can observe that the tones of the chromatic scale and the tonalities of the circle of fifths are identical. The circle of fifths then is *chromatic scale of chromatic scales*. We can expect it to behave in exactly the same manner. Since the diatonic comma appears after three moves in either direction in the chromatic scale (laid out in fifths) we can expect the circle to do the same. In the progression to the right the *entire* system of **D** is divided by the diatonic comma (**1.0125**) *before* the wheel is rotated. When the shift is made and the structure is rotated, the tonality (which, without the comma would be **.987654321**) would be multiplied by **1.0125** and would be restored to the correct value of **1.0**. The same thing would happen to the **B** (**1.875**) in the opposite column. It would be reduced to **1.851851** and then multiplied by **1.0125** to return it to **1.875**. Exactly the opposite happens when the move is to the left, between **B♭** and **E♭**.

These charts provide a better understanding of the particular nature of musical reciprocity. The square of the number at the top of any complex when divided by any number in one column will produce the correct opposite value, adjusting for the octave. Assuming **D** is at the top of the circle we can observe:

$$\frac{1.125^2}{1.875} = \frac{1.265625}{1.875} = .675$$

$$.675 * 2 = 1.35$$

Except for the placement of the commas, the **F♯** structure would be the exact reverse of the **C** structure. Numerically it would appear that the comma comes one step early in the **C** structure, however, it must do this so as to keep the reciprocal triad (**F A C**) pure. In the structure of **F♯** we have the best of both worlds. The reciprocal triad

F♯	**1.40625**	**45**
D♯	**1.171875**	**75**
B	**1.875**	**15**

is held pure within the structure while the structure itself gives us the familiar modified power series of **3**.

If we allow **45** to be the principal tone the system will admit a **B♯** of **1.97753906**. This gives us the chromatic series as we developed it earlier.

F♯	**1.40625**	**1.40625**	**F♯**
B	**1.875**	**1.05468**	**C♯**
E	**1.25**	**1.58203**	**G♯**
⊗	⊗	⊗	⊗
A	**1.6875**	**1.17187**	**D♯**
D	**1.125**	**1.75187**	**A♯**
G	**1.5**	**1.31835**	**E♯**
C	**1.0**	**1.97753**	**B♯**

45	**45**
15	**135**
5	**405**
⊗	⊗
27	**75**
9	**225**
3	**675**
1	**2025**

These are the numbers that appear in the matrix of the chromatic scale. When **F♯** is the principal tone the fundamental is **C**.

	405	**2025**
27	**135**	**675**
9	**45**	**225**
3	**15**	**75**
1	**5**	

	G♯	**B♯**
A	**C♯**	**E♯**
D	**F♯**	**A♯**
G	**B**	**A**
C	**E**	

In the more familiar structure with **C** as the principal tone the fundamental is **G♭**.

	405	**2025**
27	**135**	**675**
9	**45**	**225**
3	**15**	**75**
1	**5**	

	D	**F♯**
E♭	**G**	**B**
A♭	**C**	**E**
D♭	**F**	**A**
G♭	**B♭**	

Another method of developing the values of the circle of fifths is to multiply the basic values of the chromatic scale of **C** by each individual value, **D♭**, **D**, **E♭**, etc. The following charts illustrate this procedure, first to the right and then to the left. Included are the values represented as multiples of **xyz**. It must be remembered that just as **xy** can be **yx** so any of these structures can be interpolated. Once again we see can see that the fundamental (**1.0**) is held constant and the notes produced are the same as those produced by rotating the wheel. Multiplying in this manner also automatically make the correct placement of the commas.

Combining the **xyz** structures produces the decimal value of each structure.

$$x = 1.125$$
$$y = 1.11111$$
$$z = 1.06666$$

$$xyz\ x\ yxz = x^3 y^2 z^2 = 2$$
$$xyz\ x = x^2 yz = 1.5$$

The charts on the following two pages illustrate this.

C		G		D		A		E		B		F♯	
C	2.0	C	2.0	C	2.0	C	2.0	C	2.0	C	2.0	C	2.0
xyz x yxz		*xyz x yxz*		*xyz x yxz*		*xyz x yxz*		*xyz x yxz*		*xyz x yxz*		*xyz x yxz*	
B	1.875	B	1.875	B	1.875	B	1.875	B	1.875	B	1.875	B	1.875
xyz x yx		*xyz x yx*		*xyz x yx*		*xyz x yx*		*xyz x yz*		*xyz x yx*		*xyz x yx*	
B♭	1.77777	B♭	1.8	B♭	1.8	B♭	1.77777	A♯	1.75781	A♯	1.75781	A♯	1.75781
xyz x yz		*xyz x xz*		*xyz x xz*		*xyz x yz*		*xyx yx*		*xyx yx*		*xyx yx*	
A	1.66666	A	1.6875	A	1.6875	A	1.66666	A	1.66666	A	1.66666	A	1.6875
xyz x y		*xyz x x*		*xyz x x*		*xyz x y*		*xyz x y*		*xyz x y*		*xyz x x*	
A♭	1.6	A♭	1.6	G♯	1.58203	G♯	1.5625	G♯	1.5625	G♯	1.5625	G♯	1.58203
xyz x z		*xyz x z*		*xyx x*		*xyx y*		*xyx y*		*xyx y*		*xyx x*	
G	1.5	G	1.5	G	1.5	G	1.48148	G	1.5	G	1.5	G	1.5
xyz x		*xyz x*		*xyz x*		*xyz y*		*xyz x*		*xyz x*		*xyz x*	
F♯	1.40625	F♯	1.40625	F♯	1.40625	F♯	1.38888	F♯	1.40625	F♯	1.40625	F♯	1.40625
xyx		*xyx*		*xyx*		*xyy*		*xyx*		*xyx*		*xyx*	
F	1.33333	F	1.33333	F	1.35	F	1.33333	F	1.33333	E♯	1.31835	E♯	1.31835
xyz		*xyz*		*xzx*		*xyz*		*xyz*		$x\,{}^{y}/_{z}\,x$		$x\,{}^{y}/_{z}\,x$	
E	1.25	E	1.25	E	1.26526	E	1.25	E	1.25	E	1.25	E	1.25
xy		*xy*		*xx*		*xy*		*xy*		*xy*		*xy*	
E♭	1.2	E♭	1.2	E♭	1.2	D♯	1.17187	D♯	1.17187	D♯	1.17187	D♯	1.17187
xz		*xz*		*xz*		${}^{xy}/_{z}$		${}^{xy}/_{z}$		${}^{xy}/_{z}$		${}^{xy}/_{z}$	
D	1.125	D	1.125	D	1.125	D	1.11111	D	1.11111	D	1.125	D	1.125
x		*x*		*x*		*y*		*y*		*x*		*x*	
D♭	1.06666	D♭	1.06666	C♯	1.05468	C♯	1.04166	C♯	1.04166	C♯	1.05468	C♯	1.05468
z		*z*		${}^{x}/_{z}$		${}^{y}/_{z}$		${}^{y}/_{z}$		${}^{x}/_{z}$		${}^{x}/_{z}$	
C	1.0	C	1.0	C	1.0	C	1.0	C	1.0	C	1.0	C	1.0

C		F		B♭		E♭		A♭		D♭		G♭	
C	2.0	C	2.0	C	2.0	C	2.0	C	2.0	C	2.0	C	2.0
xyz x yxz		*xyz x yxz*		*xyz x yxz*		*xyz x yxz*		*xyz x yxz*		*xyz x yxz*		*xyz x yxz*	
B	1.875	B	1.875	C♭	1.89629	C♭	1.92	C♭	1.92	C♭	1.89629	C♭	1.89629
xyz x yx		*xyz x yx*		*xyz x zyz*		*xyz x zxz*		*xyz x zxz*		*xyz x zyz*		*xyz x zyz*	
B♭	1.77777	B♭	1.77777	B♭	1.77777	B♭	1.8	B♭	1.8	B♭	1.77777	B♭	1.77777
xyz x yz		*xyz x yz*		*xyz x yz*		*xyz x xz*		*xyz x yz*		*xyz x yz*		*xyz x yz*	
A	1.66666	A	1.66666	A	1.66666	A	1.6875	B♭♭	1.70666	B♭♭	1.70666	B♭♭	1.70666
xyz x y		*xyz x y*		*xyz x y*		*xyz x x*		*xyz z xz*		*xyz z xz*		*xyz z xz*	
A♭	1.6	A♭	1.6	A♭	1.58024	A♭	1.6	A♭	1.6	A♭	1.6	A♭	1.6
xyz x z		*xyz x z*		*xyz y z*		*xyz x z*		*xyz x z*		*xyz x z*		*xyz x z*	
G	1.5	G	1.5	G	1.48148	G	1.5	G	1.5	G	1.5	A♭♭	1.51703
xyz x		*xyz x*		*xyz y*		*xyz x*		*xyz x*		*xyz x*		*zyz xz*	
F♯	1.40625	G♭	1.42222	G♭	1.42222	G♭	1.44	G♭	1.42222	G♭	1.42222	G♭	1.42222
xyx		*x zyz*		*x zyz*		*x zxz*		*x zyz*		*x zyz*		*x zyz*	
F	1.33333	F	1.33333	F	1.33333	F	1.35	F	1.33333	F	1.33333	F	1.33333
xyz		*xyz*		*xyz*		*xzx*		*xyz*		*xyz*		*xyz*	
E	1.25	E	1.25	E	1.25	F♭	1.28	F♭	1.28	F♭	1.28	F♭	1.26419
xy		*xy*		*xy*		*zxz*		*zxz*		*zxz*		*xyx*	
E♭	1.2	E♭	1.18518	E♭	1.18518	E♭	1.2	E♭	1.2	E♭	1.2	E♭	1.18518
xz		*yz*		*yz*		*xz*		*xz*		*xz*		*yz*	
D	1.125	D	1.11111	D	1.11111	D	1.125	D	1.125	E♭♭	1.1377	E♭♭	1.1377
x		*y*		*y*		*x*		*x*		*zz*		*zz*	
D♭	1.06666	D♭	1.06666	D♭	1.06666	D♭	1.06666	D♭	1.06666	D♭	1.06666	D♭	1.06666
z		*z*		*z*		*z*		*z*		*z*		*z*	
C	1.0	C	1.0	C	1.0	C	1.0	C	1.0	C	1.0	C	1.0

The preceding charts represent the complete enharmonic scale. No further development will be needed. To briefly summarize, we constructed first the diatonic scale with two alternate notes, creating, in effect, three parallel diatonic scales. This trinity allows us to maintain proper tuning in the tonality of the center.

		C	**2.0**		
		B	**1.875**		
		A	**1.666666**	**A**	**1.6875**
		G	**1.5**		
		F	**1.33333**		
		E	**1.25**		
D	**1.111111**	**D**	**1.125**		
		C	**1.0**		

This, in effect, creates a nine tone functional scale. Using the same procedure we constructed the tritonal chromatic scale. This doubles the number of scale members and provides enharmonic alternates for all of the intervals which are not perfect as well as tritonal alternates as with the diatonic scale.

		C	**2.0**		
		B	**1.875**		
		B♭	**1.777777**	**B♭**	**1.8**
		A	**1.666666**	**A**	**1.6875**
		A♭	**1.6**		
		G	**1.5**		
G♭	**1.4222222**	**F♯**	**1.40625**		
		F	**1.333333**		
		E	**1.25**		
E♭	**1.185185**	**E♭**	**1.2**		
D	**1.111111**	**D**	**1.125**		
		D♭	**1.0666666**	**C♯**	**1.0546875**
		C	**1.0**		

By continuing the progression of parallel tonalities, each a fifth apart, we developed the enharmonic scale. This scale is, functionally, thirteen chromatic scales, creating a linear thirty-seven-tone scale.

				C	**2.0**				
C♭	**1.92**	**C♭**	**1.89629**	**B**	**1.875**				
				B♭	**1.77777**	**B♭**	**1.8**	**A♯**	**1.75781**
		B♭♭	**1.70666**	**A**	**1.66666**	**A**	**1.6875**		
		A♭	**1.58024**	**A♭**	**1.6**	**G♯**	**1.52803**	**G♯**	**1.5625**
		A♭♭	**1.51703**	**G**	**1.5**	**G**	**1.48148**		
G♭	**1.44**	**G♭**	**1.42222**	**F♯**	**1.40625**	**F♯**	**1.38888**		
		F	**1.35**	**F**	**1.33333**	**E♯**	**1.31835**		
F♭	**1.26419**	**F♭**	**1.28**	**E**	**1.25**	**E**	**1.26526**		
		E♭	**1.18518**	**E♭**	**1.2**	**D♯**	**1.17187**		
E♭♭	**1.13777**	**D**	**1.11111**	**D**	**1.125**				
				D♭	**1.06666**	**C♯**	**1.05468**	**C♯**	**1.04166**
				C	**1.0**				

The scale itself will only have value if the various ratios can be expressed integrally. We achieved this in the chromatic scale by letting the tonality be equal to **45** (taking our cue from the fact that the tonality of the diatonic scale is **3**). We were then able to express the chromatic scale as integers of a precise matrix, progressing by powers of three on one axis and powers of five on the other.

(81)	**405**	**2025**		B♭	**D**	**F♯**
27	**135**	**675**		E♭	**G**	**B**
9	**-45-**	**225**		A♭	**-C-**	**E**
3	**15**	**75**		D♭	**F**	**A**
1	**5**	25		G♭	B♭	D

If we imagine a line drawn diagonally through **1**, **45** and **2025** we see the power series of **45** developing. This is superpositioning in tritones. **2025** is 45^2 and the point of reciprocity of the structure. **2025** divided by any number on one side of the line produces its reciprocal opposite. The structure can be thought of as six reciprocal pairs plus **45**. (**1 - 2025, 3 - 675, 9 - 225, 27 - 75, 5 - 405, 15 - 135, 45 -45**)

Since **45** is the 'top' of the diatonic scale which is nine tones and **2025** or 45^2 is the 'top' of the chromatic scale of eighteen tones we can expect 2025^2 or **4100625**, to be the 'top' of the enharmonic scale of thirty-seven tones. This would produce a matrix of 3^8 (**6561**) on the vertical and 5^4 (625) on the horizontal. This is as large as the matrix needs to get for all practical musical purposes.

45^2 produces the augmented seventh which is the inversion of the enharmonic comma. This structure can be considered to be a superposition of the enharmonic comma.

$$\frac{45^2}{2^{10}} = \frac{2025}{1024} = 1.9775390625$$

$$\frac{2}{1.9775390625} = 1.01135802469$$

2025^2 (45^4) misses 2^{22} by the square of the enharmonic comma.

$$\frac{2^{22}}{2025^2} = \frac{4194304}{4100625} = 1.0228450541076$$

$$\sqrt{1.0228450541076} = 1.01135802469$$

The fundamental is still **1** but the tonality would become **2025**. We discovered previously that if we multiply the values of the chromatic scale by $45 * 2^5 = 1440$ we produce the scale as integers of one octave

1440	1440
1920	2160
2560	1620
⊗	⊗
1728	2400
2304	1800
1536	2700
2048	2025

	1620	2025
1728	2160	2700
2304	1440	1800
1536	1920	2400
2048	2560	

1.0	1.0
1.3333	1.5
1.7777	1.125
⊗	⊗
1.2	1.6666
1.6	1.25
1.0666	1.875
1.4222	1.4062

If we multiply the values of the enharmonic scale as they exist in the prime octave $2/1$ by 1036800 $(2025 * 2^9)$ we produce the enharmonic scale as integers of one octave.

		1.011358 a			
F♯	1.40625	1458000	G♭	1.422222	1474560
	1.0125	a		1.0125	a
F♯	1.388888	1440000	G♭	1.44	1492992
	1.0208806	b		1.0208806	b
F	1.35	1399680	G	1.481481	1536000
	1.0125	a		1.0125	a
F	1.333333	1382400	G	1.5	1555200
	1.011358	c		1.0125	c
E♯	1.31839	1366875	A♭	1.51703	1572864
	1.029968	d		1.029968	d
F♭	1.28	1327104	G♯	1.5625	1620000
	1.011358	c		1.011358	c
E	1.265625	1312200	A♭	1.58024	1638400
	1.0011295	e		1.0011295	e
F♭	1.26419	1310720	G♯	1.58203	1640250
	1.011358	c		1.011358	c
E	1.25	1296000	A♭	1.6	1658880
	1.041666	f		1.041666	f
E♭	1.2	1244160	A	1.666666	1728000
	1.0125	a		1.0125	a
E♭	1.185185	1228800	A	1.6875	1749600
	1.011358	c		1.011358	c
D♯	1.171875	1215000	B♭♭	1.706666	1769472
	1.029968	d		1.029968	d
E♭♭	1.1377777	1179648	A♯	1.75781	1822500
	1.011358	c		1.011358	c
D	1.125	1166400	B♭	1.777777	1843200
	1.0125	a		1.0125	a
D	1.1111111	1152000	B♭	1.8	1866240
	1.041666	f		1.041666	f
D♭	1.0666666	1105920	B	1.875	1944000
	1.011358	c		1.011358	c
C♯	1.0546875	1093500	C♭	1.89629	1966080
	1.0125	a		1.0125	a
C♯	1.0416666	1080000	C♭	1.92	1990656
	1.041666	f		1.041666	f
C	1.0	1036800	C	2.0	2073600

Each of the prime octave decimals is represented by a whole number and this series is the smallest series of integers that can represent the enharmonic scale. The enharmonic scale is totally reciprocal. Each column is identical. The scale is composed of six discrete intervals.

a	**1.0125**
b	**1.028806**
c	**1.011358**
d	**1.029968**
e	**1.0011295**
f	**1.0416666**

If we arrange these in accordance with the groupings defined by the **xyz** structure of the diatonic and chromatic scales we get.

facfa cdcaf cecdc aba c aba cdcec facdc afcaf

The enharmonic scale is a totally complete music system. It needs no further development. This scale is balanced, symmetrical and can be considered a closed system.

THE ENHARMONIC MATRIX

The integers of the enharmonic scale as shown are upper octaves of odd numbers with the exception of **E♯** (**1.318359375**), which numerically is **1366785.** If we reduce these numbers to their basic octaves, that is, the value at which they enter the system, we get.

1458000	**1474560**
1440000	**1492992**
1399680	**1536000**
1382400	**1555200**
1366875	**1572864**
1327104	**1620000**
1312200	**1638400**
1310720	**1640250**
1296000	**1658880**
1244160	**1728000**
1228800	**1749600**
1215000	**1769472**
1179648	**1822500**
1166400	**1843200**
1152000	**1866240**
1105920	**1944000**
1093500	**1966080**
1080000	**1990656**
1036800	**2073600**
↑	↓

91125	**45**
5625	**729**
10935	**375**
675	**6075**
1366875	**3**
81	**50625**
164025	**25**
5	**820125**
10125	**405**
1215	**3375**
75	**54675**
151875	**27**
9	**455625**
18225	**225**
1125	**3645**
135	**30375**
273375	**15**
16875	**243**
2025	**2025**
↑	↓

Once again we have a series of numbers that, as with the chromatic scale, create a matrix.

If we arrange these, letting **C=2025**, in the form of the matrix developed by the powers of three on one axis and five on the other, we get the system illustrated below. Unlike the chromatic matrix, however, neither the fundamental (**1**) nor the point of reciprocity (in this case **4100625**) appears in the functional part of the matrix.

		164025	**820125**	
	10935	**54675**	**273375**	**13366875**
729	**3645**	**18225**	**91125**	**455625**
243	**1215**	**6075**	**30375**	**151875**
81	**405**	**2025**	**10125**	**50625**
27	**135**	**675**	**3375**	**16875**
9	**45**	**225**	**1125**	**5625**
3	**15**	**75**	**375**	
	5	**25**		

		E	**G♯**	
	F	**A**	**C♯**	**E♯**
G♭	**B♭**	**D**	**F♯**	**A♯**
C♭	**E♭**	**G**	**B**	**D♯**
F♭	**A♭**	**C**	**E**	**G♯**
B♭♭	**D♭**	**F**	**A**	**C♯**
E♭♭	**G♭**	**B♭**	**D**	**F♯**
A♭♭	**C♭**	**E♭**	**G**	
	F♭	**A♭**		

The chart on the following page presents the matrix with both the prime octave decimals and the integral values. The chromatic scale is formed between the tritone (**45**) and its reciprocal inversion. The enharmonic scale is formed between two enharmonic values of the principal tone, which is now **2025**. The fundamental (**1**) is **D♭♭** (**1.01135802**) and the reciprocal point is **B♯** (**1.97753906**). The values of the principal tone **C** altered by the diatonic comma (**1.0125**) also lie beyond the area defined by the enharmonic scale. Thus **C**, the principal tone or tonality of the system is, as it must be, unchallenged.

Counting the fundamental (**1**) and its enharmonic equivalent (**4100625**) the system covers twenty-one octaves. The actual system is nineteen octaves. This is a very large system considering the range of human hearing is about ten octaves.

6561 A♭ 1.62	320805 C 1.0125	**164025 E 1.265625**	**820125 G♯ 1.528031**	4100625 B♯ 1.9775391
2187 D♭ 1.08	**10935 F 1.35**	**54675 A 1.6875**	**273375 C♯ 1.0546875**	**1366875 E♯ 1.318393**
729 G♭ 1.44	**3645 B♭ 1.8**	**18225 D 1.125**	**91125 F♯ 1.40625**	**455625 A♯ 1.7578125**
243 C♭ 1.92	**1215 E♭ 1.2**	**6075 G 1.5**	**30375 B 1.875**	**151875 D♯ 1.171875**
81 F♭ 1.28	**405 A♭ 1.6**	**2025 C 1.0 (2.0)**	**10125 E 1.25**	**50625 G♯ 1.5625**
27 B♭♭ 1.7066666	**135 D♭ 1.0666666**	**675 F 1.33333333**	**3375 A 1.66666666**	**16875 C♯ 1.04166666**
9 E♭♭ 1.13777777	**45 G♭ 1.4222222**	**225 B♭ 1.7777777**	**1125 D 1.11111111**	**5625 F♯ 1.38888888**
3 A♭♭ 1.517037	**15 C♭ 1.896296**	**75 E♭ 1.185185**	**375 G 1.481481**	1875 B 1.851851
1 D♭♭ 1.011358	**5 F♭ 1.2641975**	**25 A♭ 1.5802469**	125 C 1.975308	625 E 1.234567

We can now observe the following progressions.

6561	*32805*	**164025**	**820125**	*4100625*
2187	**10935**	**54687**	**273375**	**13366875**
729	**3645**	**18225**	**91125**	**455625**
243	**1215**	**6075**	**30375**	**151875**
81	**405**	**2025**	**10125**	**50625**
27	**135**	**675**	**3375**	**16875**
9	**45**	**225**	**1125**	**5625**
3	**15**	**75**	**375**	*1825*
1	**5**	**25**	*125*	*625*

A	C	**E**	**G♯**	**B**
D♭	**F**	**A**	**C♯**	**E♯**
G♭	**B♭**	**D**	**F♯**	**A♯**
C♭	**E♭**	**G**	**B**	**D♯**
F♭	**A♭**	**C**	**E**	**G♯**
B♭♭	**D♭**	**F**	**A**	**C♯**
E♭♭	**G♭**	**B♭**	**D**	**F♯**
A♭♭	**C♭**	**E♭**	**G**	**B**
D♭♭	**F♭**	**A♭**	C	*E*

The line of the commas runs from right to left. It enters the matrix at **125**, forming **1.024 (128/125 = 1.024)**. Next it touches **2025**, forming **1.01135802 (2048/2045 = 1.01135802).** It finally quits the matrix at **32805**, forming **1.00112915 (32768/32805 = 1.00112915)**. The relationship of each progression is the same as that of **5** and **81** establishing the fact that each successive comma is a modification of **1.024** by **1.0125.**

The other line, going from left to right, shows the numerical center of the matrix (**2025**), which is also the power series of **45**. The reciprocal point in this case is **4100625**. This number divided by any number on one side of the line produces its positional counterpart on the other side of the line.

$$\frac{4100625}{125} = 32805 \quad \frac{4100625}{32805} = 125 \quad \text{etc.}$$

If we discount the principal tone (**2025**) there are eighteen intervals and their inversions. Thirteen of these intervals are grouped as right base generations of **2025** itself, that is, if we let **2025=1** these tones can still be represented by whole numbers.

The result is the familiar chromatic matrix, shown here with its inversions.

		164025	**820125**	
		54687	**273375**	**1366875**
		18225	**91125**	**455625**
		6075	**30375**	**151875**
81	405	**2025**	**10125**	**50625**
27	135	675		
9	45	225		
3	15	75		
	5	25		

		E	**G♯**	
		A	**C♯**	**E♯**
		D	**F♯**	**A♯**
		G	**B**	**D♯**
F♭	A♭	**C**	**E**	**G♯**
B♭♭	D♭	F		
E♭♭	G♭	B♭		
A♭♭	C♭	E♭		
	F♭	A♭		

If **C=2025** then this is a descending chromatic scale of **F♯** on the upper right and its reciprocal, an ascending chromatic scale of **G♭** on the lower left. There are ten intervals remaining (five reciprocal pairs). These are formed by various octaves of the power series of **3** and various octaves of the power series of **5**.

Since neither member of the ratio is an octave of the fundamental (the power series of **2**) the notion of root position becomes slightly more oblique. Let us assume that because of the acoustical power of the power series of **3** we can expect that an interval formed by powers of **5** and powers of **3** can be said to be in root position when a power of **3** is in the denominator. Thus $^{\mathbf{5}}/_{\mathbf{3}}$ is the root position of **A** (**1.6666**) even though its inversion $^{\mathbf{6}}/_{\mathbf{5}}$ is a prime interval.

These ten new values fill out the matrix.

	(32805)	**164025**	**820125**	
	10935	**54687**	**273375**	**1366875**
729	3645	**18225**	**91125**	**455625**
243	1215	**6075**	**30375**	**151875**
81	405	**2025**	**10125**	**50625**
27	135	675	**3375**	**16875**
9	45	225	**1125**	**5625**
3	15	75	**375**	
	5	25	*(125)*	

	(C)	**E**	**G♯**	
	F	**A**	**C♯**	**E♯**
G♭	B♭	**D**	**F♯**	**A♯**
C♭	E♭	**G**	**B**	**D♯**
F♭	A♭	**C**	**E**	**G♯**
B♭♭	D♭	F	**A**	**C♯**
E♭♭	G♭	B♭	**D**	**F♯**
A♭♭	C♭	E♭	**G**	
	F♭	A♭	*(C)*	

Now the line of the octave, **125 - 2025 - 32805**, forms the reciprocal structure. This is produced by what might be called the regression of the commas. If the notes are represented by integers then **4100625** divided by any integer in the matrix produces its inversion. If the notes are represented by prime octave decimals then **2** divided by any of the decimals in the matrix produces its inversion. In the non-linear mathematics of music (and as we shall see, vibration) **1** and **2** have the same relative position to one another as any integer and its square.

If we now illustrate the enharmonic matrix as the ratios that produce the prime octave values we can observe why, structurally, **2025** is the center of the system even thought **1** remains the absolute fundamental. For practical reasons we consider the enharmonic scale as being composed of integers. Structurally we must consider it as being composed by ratios.

Each decimal value in bold type, (**G=1.5, D=1.125**, etc.) is produced by an odd integer divided by a power of **2**. The reciprocals on the other hand, are produced by a power **2** divided by the same odd integers. **F♯**, or **1.40625** would be $^{45}/_{32}$. Its inversion would be **G♭**, **1.422222** or $^{64}/_{45}$.

		$\frac{81}{64}$	$\frac{405}{256}$	
		E 1.265625	G♯ 1.528031	
	$\frac{27}{20}$	$\frac{27}{16}$	$\frac{135}{128}$	$\frac{675}{512}$
	F 1.35	A 1.6875	C♯ 1.0546875	E♯ 1.318393
$\frac{36}{25}$	$\frac{9}{5}$	$\frac{9}{8}$	$\frac{45}{32}$	$\frac{225}{128}$
G♭ 1.44	B♭ 1.8	D 1.125	F♯ 1.40625	A♯ 1.7578125
$\frac{48}{25}$	$\frac{6}{5}$	$\frac{3}{2}$	$\frac{15}{8}$	$\frac{75}{64}$
C♭ 1.92	E♭ 1.2	G 1.5	B 1.875	D♯ 1.171875
$\frac{32}{25}$	$\frac{8}{5}$	$\frac{2}{1}$	$\frac{5}{4}$	$\frac{25}{16}$
F♭ 1.28	A♭ 1.6	C 1.0 (2.0)	E 1.25	G♯ 1.5625
$\frac{128}{75}$	$\frac{16}{15}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{25}{24}$
B♭♭ 1.706666	D♭ 1.066666	F 1.333333	A 1.666666	C♯ 1.041666
$\frac{256}{225}$	$\frac{64}{45}$	$\frac{16}{9}$	$\frac{10}{9}$	$\frac{25}{18}$
E♭♭ 1.137777	G♭ 1.422222	B♭ 1.777777	D 1.111111	F♯ 1.388888
$\frac{1024}{675}$	$\frac{256}{135}$	$\frac{32}{27}$	$\frac{40}{27}$	
A♭♭ 1.517037	C♭ 1.896296	E♭ 1.185185	G 1.481481	
	$\frac{512}{405}$	$\frac{128}{81}$		
	F♭ 1.2641975	A♭ 1.5802469		

SUMMARY

The enharmonic scale is a precise and discrete reality. There is one, and only one enharmonic scale and the definition offered in Part I is an absolute. The enharmonic scale is easy to construct, as it is the product of only three numbers, **3**, **4** and **5**.

Music is constructed functionally by a series of prime ratios, that is, ratios comprised of consecutive numbers. The prime interval is the ideal in music and the preceding scale allows us to fully realize this ideal.

These ratios, or intervals, form a series that we shall see appear in various aspects of acoustics and vibration.

$\frac{81}{80}$	**1.0125**
$\frac{25}{24}$	**1.04166666**
$\frac{16}{15}$	**1.06666666**
$\frac{10}{9}$	**1.11111111**
$\frac{9}{8}$	**1.125**
$\frac{6}{5}$	**1.2**
$\frac{5}{4}$	**1.25**
$\frac{4}{3}$	**1.33333333**
$\frac{3}{2}$	**1.5**
$\frac{2}{1}$	**2.0**

We will seldom see it progress past $16/15$ but it will be seen often. Those seven ratios between $3/2$ and $16/15$ (inclusive) plus their inversions provide sufficient material to create the pentatonic, diatonic and enharmonic scales. That is to say that every scale member of these three scales will form an interval with the tonality that is either a prime interval or its inversion.

We have seen that to maintain the integrity of these ratios we had to construct a progressive series of thirteen interrelated scales. These are structured by fifths from a given tone (**1.422222**) to its enharmonic alternate (**1.40625**). These fifths are modified after each set of three progressive fifths three times by the diatonic comma (**1.0125**) and once by the chromatic comma (**1.024**). The two commas are concurrent after the sixth member of the series. If the move is to the right the diatonic comma lowers the value of the affected fifth $1.5/1.0125 = 1.481481481$. The chromatic comma would raise the same fifth $1.5 * 1.024 = 1.536$. The chromatic comma divided by the diatonic comma produces the comma we actually observe at this midpoint of the structure.

$$1.024 / 1.0125 = 1.01135802469$$

By the same token the cube of the diatonic comma divided by the chromatic comma produces the Pythagorean Comma.

$$\frac{1.0125^3}{1.024} = 1.0136432647705078125$$

This ability to tune by shifting tonalities structurally within a single tonal center allows us to displace the various commas linearly. By this method the fifths, fourths and major thirds (and their inversions) are always the ideal ratios.

The differential tones created by intervals of the various tonalities create, in turn, simple, hierarchical structures with various sub-fundamentals. The conglomerate of these, in turn, produces a grand fundamental. Because of this it is important to realize that the differential tone is *real*. It is in every way just as real as the tones that create it.

Playing in tune then is nothing more than keeping the overall differential structure as simple as is mathematically possible. Playing in tune, however, is only a small part of what this device will be capable of ultimately providing. We can observe that if we trace the ultimate structure of any given group of inputs we arrive at a simple arithmetic overtone structure, **1, 2, 3, 4** … etc. It is from this overtone series that the structure, which produces the tone itself, arises.

Music is a specific form of sound. Physically speaking it is nothing else. All sound is the result of vibration. All vibration is similar, no matter if it is the vibration of an atom, the sound of a trombone or the ring of the earth caused by an earthquake.

Newtonian physics tells us that if we reduce the mass of a vibrating body by $1/2$, the fundamental (and thus its structure) is raised in pitch by one octave. How small must the mass become before this cease to be true? If this is not the atom itself it can't be any larger that a crystalline structure. On the other hand how large can we make this vibrating mass? There would appear to be no practical limit. It could range form the atom to the universe itself.

There is no reason to believe that there would be any inconsistency in the nature of vibration, no matter what the mass of the vibrating body. What is true for one size would be true for all.

Because of its special nature music offers us a look at this energy restructuring we call vibration in a very unique manner. The mathematics of Part I is complete. This mathematics will provide us with the ability to exert a control over a vibrating, or stationary structure that will be truly awesome. This control is precise, digital and absolute.

As we understand more of the nature of a vibrating structure we will see that the ability to keep our scales in tune is truly a secondary function of the enharmonic system.

The mathematics of the enharmonic system is useful only to the degree that they can be applied empirically to physical reality. This will be realized in the following sections.

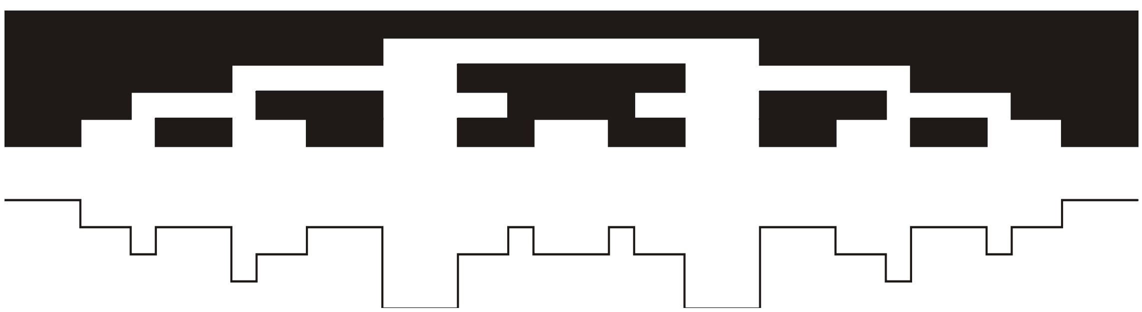
STRUCTURAL RESONANCE

THE MATHEMATICS OF MUSIC AND VIBRATION

PART TWO
THE NON-LINEAR WAVE

By

Thomas Wagner



ALL THE SOUNDS OF THE EARTH ARE LIKE MUSIC
Oscar Hammerstein

Mathematics deals exclusively with the relation of concepts to each other without consideration of their relation to experience. Physics too deals with mathematical concepts; however, these concepts attain physical content by the clear determination of their relation to the objects of experience.
Albert Einstein

Third time's a charm! A broken mirror brings *seven* years bad luck. A cat has *nine* lives. *Seven* come *eleven* and Friday the *thirteenth*. The numbers of magic? Of numerology? Of children's rhymes and ancient folklore? Yes, all of these but they are also the numbers of music, or, more correct, the numbers that aren't part of music. A common theme runs through ancient writings, especially those of the Hellenes. This states that good is associated with the rational numbers whereas 'evil comes in through the irrational'.

It seems numbers have always intrigued man. The inquisition actually burned those who held to the Pythagorean concept of the number three. The Pythagoreans held the number three to be the perfect number because it had a beginning, middle and an end. The inquisition, on the other hand, held the number three to be the perfect number because it had no beginning middle or end. What could number possibly have meant to the medieval church that would incite such a reaction?

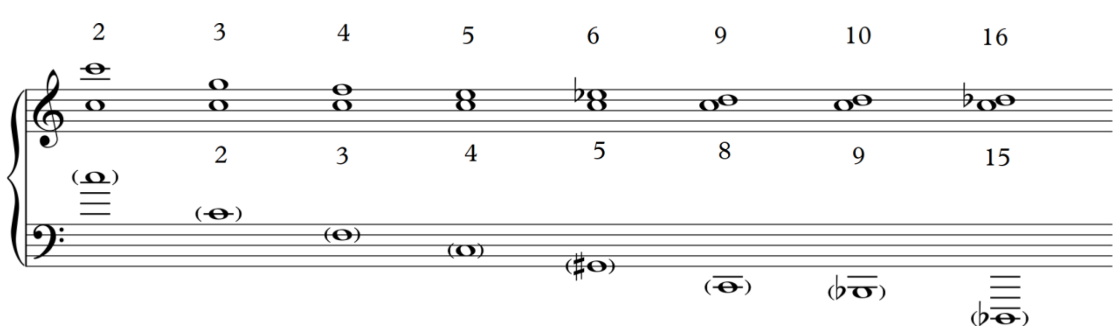
The number three is associated with the circle. When musical calligraphy first developed triple time was considered to be the perfect time signature. The symbol used to designate this was the circle. A trace of that remains today as four-four time or so-called common time indicated by a broken circle or C. The circle was broken to indicate imperfect time.

Part one offers a comprehensive mathematical description of music phenomenon. It is important to understand what music is in the physical world in order to put these numbers to a logical use. Here again we will find that all aspects of music exist as a complex arrangement of one basic phenomenon, the differential tone. Understand the differential tone and one understands all of music.

The differential tone is a real tone created by sounding at least two frequencies together. In fact, most of what we consider to be music is the result of things that happens somewhere else in the system. We shall discover that what we call timbre is the result of a complex differential structure. It is the differential tone that gives the perfect fifth its power (a phenomenon as real today as it was to Cro Magnon).

It is impossible to sound two or more distinct frequencies without producing at least one differential tone, and in most cases, several of them. Making music is really nothing more than a precise and controlled modification of the differential structure. There can only be one fundamental. No matter what complex of frequencies is sounded, one, and only one, fundamental will appear.

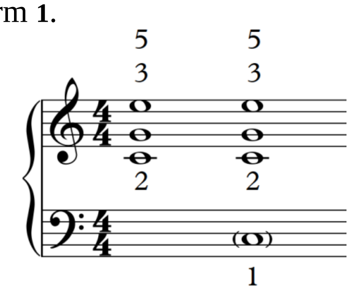
Here are the primary differential tones produced by the hierarchy of prime intervals. As we increase the values of the two numbers forming the prime interval the size of the interval becomes smaller and the 'distance' to the fundamental is increased. Thus far in our observations we have held the fundamental as a constant allowing the intervals themselves to occupy their normal overtone positions. Let us look at these intervals as they would appear in music theory, that is, with the bottom note of the interval held as a constant.



As the interval become smaller the differentially produced fundamental becomes more remote. For the current discussion it is important that we are aware of two things: a., in every case but the octave the fundamental is a separate tone differentially produced by the two sounding tones, and b., the fundamental does not enter into differential relationships with any other tone. The fundamental includes every sounding ratio because it is created by those very ratios.

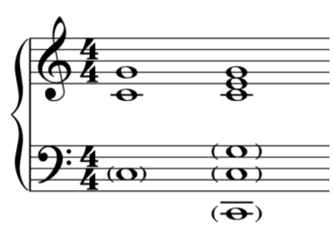
It is for this reason that prime intervals are especially significant. Since the fundamental is a non-participatory member of the structure (that is, it forms no differential tones with any other note of the interval) it follows that a prime interval adds nothing to the structure but a boost to the fundamental (which is of course the ultimate goal of all concerted music). This is true regardless of the size of the numbers that create the prime interval. $81/80$ is just as benign to the system as $5/4$. Any ratio that is not created by consecutive integers will form a differential tone that is not the fundamental and thus must be considered to be an active member of the integral structure that will ultimately create the fundamental.

Consider the interval $6/5$, the minor third. Because of the fact that it contains an even number this interval is the inversion of a more basic interval, that is, $5/3$, the ideal major sixth. The primary differential tone formed by this interval is **2**. While **1**=**2** and thus **2** is in every respect the 'same' as the fundamental the fact remains that **2** is an active member of the structure while **1** is not. **2** exists as a real tone and thus, as we have seen previously by example, **2** will react with **3** to form **1**.



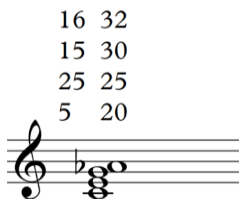
This $5/3$ forms two differential tones **1** and **2**. All non-prime intervals behave in this manner. Notice that any given interval has the same fundamental as its inversion.

We can observe that any increase of the complexity of an input structure lowers the fundamental. Consider the following example.



If nothing else is sounding, then the perfect fifth is always $3/2$ and produces, differentially, nothing but the octave below **2**. On the other hand, if we add the major third above the lower tone we create the triad. In this case the fifth must be $6/4$ instead of $3/2$. The interval $\frac{6}{4}$ produces, of course, **2** as its primary differential tone. **2** is an active tone and will react with **3** to form **1**, further increasing the amplitude of the fundamental. The addition of the third lowers the fundamental by one octave.

It is, of course, not always that simple. Consider a common, somewhat modern structure.

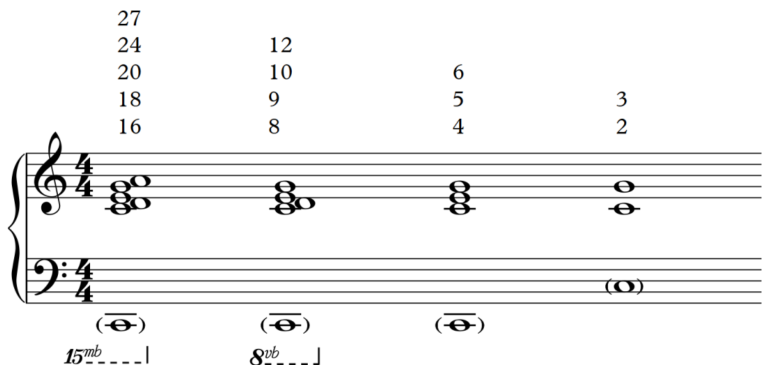


The simple addition of **A♭** completely changes the numerical structure. What was **4,5,6** now becomes **20,25,30,32**. The fundamental of this structure now lies two octaves and a major third below the fundamental of the simpler triad and is a low **A♭**. However, because of the position of the very powerful triad the ear would hear and accept **5** as the fundamental. We would class it as a C triad with a lowered or diminished sixth even though the true fundamental is **A♭.**

The triadic structure would cause **5** to be much louder than **1. 5** also has the advantage of lying more than two octaves nearer the complex itself. As long as the triad is sounding alone the only differential tones possible are **1** and **2** (**5** and **10** in the case of the four tone structure). When the minor sixth (**32**) is added the number of possible differential tones increases greatly. **5**, since it is no longer the fundamental, becomes an active member of the series. It reacts with **32** to form **27. 5** and **27** are both real tones and, because of the strength if **5**, they are very prominent**. 25** sounds with **27** to form **2**, which is fine. **27** reacts with **30** to form **3**, also fine. **20** on the other hand reacts with **27** to form the troublesome **7**. **25** reacts with **32** to also form **7**, thus increasing the amplitude of **7**. **7** in turn reacts with **20** to form **13** and so on. A simple four-note input structure creates a differential structure of amazing complexity. The tiniest change of the amplitude of any input also alters this differential tone complex profoundly.

This explains at least one reason why the pentatonic scale is so successful. This scale, which is **16, 18, 20, 24, 27**, permits a very complex potential differential structure. If the entire scale sounds at once, the fundamental will lie exactly four octaves below the root of the scale (**16**). If we omit **27** from the complex it reduces to **8, 9, 10, 12,** placing the fundamental three octaves below the root of the scale. If we omit **9** the complex reduces to **4, 5, 6** bringing the fundamental to two octaves below the root. If we then omit **5** we obtain **2, 3** and the fundamental comes to within an octave.

Sounding various combinations of the scale produces variously 'colored' versions of the fundamental in various octaves.



We can see that as the fundamental becomes more remote from the structure initiating it, the differential structure can and does become much more varied and complex. A basic arithmetical series of real frequencies is formed with differing intensities. These differences in amplitude are caused by the number of combinations that in one way or another create the frequency in question by differentiation. This increase in amplitude appears to be simply additive, that is, if two intervals create the same differential tone, and with the same intensity, the resultant will have twice the amplitude. In this manner a structure is formed in which the amplitudes of the various members have nothing at all to do with their respective numerical positions in the series. Harmonic analysis shows this clearly. Sometimes a partial will have greater amplitude than the fundamental.

The structure formed will ultimately consist of all the integers from **1** to the input structure itself. The differential structure appears as though it is formed from consecutive integers. We know that it has been formed from a structure created by a very few inputs and with very limited choices.

While all of this forms a good basis for any argument in music theory what is important for this discussion at this point is the fact that *the differential structure that ultimately creates a fundamental is precisely the same, in every way, as the structure that creates what we call the timbre of a musical tone.*

If the sequential overtone structure of the tone is the same as that of the harmonic fundamental, that is, being composed of sequential frequencies with amplitudes that look to be completely random, then we must conclude that in all probability the input structures that create the fundamentals and the input structures that create tones must be, by and large, identical. With this in mind let us take a look at the structure of a musical tone.

THE TONE

In every field of endeavor there are ideas or concepts that have become dogma merely by their having been around so long. One might say that such concepts have become axiomatic by persistence. Such an idea is the notion that a tone is comprised of a fundamental and its overtones. When we analyze a sound by accepted methods that is what we see. In the preceding paper I stressed the fact that even with state-of-the-art equipment we are unable to create a sound that is real. We shall discover that the problems encountered when attempting to create a real-sounding music tone are basically the same as those found in tempered tuning.

If this is true then differentiation must play the same role in the creation of a tone as it does in the sounding of any interval. Harmonic analysis shows us that the overtone series is comprised of overtones of varying intensities. If these tones are created by super positioning, and in this case common belief is that a vibrating body is sounding all of these tones at the same time, we should expect that the overtone series would be comprised of overtones whose amplitudes grow progressively weaker as we ascend the series.

This, of course, does not happen. If it did the sound would be dreadful. Sounding a series of overtones that diminish asymptotically creates a ghastly, unmusical sound. Sometimes the actual sounding series of an instrument has missing overtones. The slightest alteration of how an instrument is played greatly affects the series. We have to question the notion of super-positioning .The overtone series is discrete and yet we use analog methods to analyze it. A Fourier transform reduces a complex wave into individual sine waves. As such it is very useful tool, but there is no reason to think that this is what happens in the function of a vibrating or resonating body,

The notion of a sequential overtone structure has its roots in the famous experiments by Pythagoras. These experiments do indicate that the integral of the frequency is a result of a reduction in mass equal to the reciprocal of the integer of the frequency. If one stops a string one third of the way from the end it will create a sound whose frequency will be three times that of the fundamental. This establishes the relationship between mass and frequency but it does nothing to establish a sequential overtone structure.

The real significance of the third overtone in the structure will become clearer as we progress. What we need is empirical examples of exactly what this input structure is. This cannot at this time be gleaned from the differential structure itself as observed in an oscilloscope. We must find this structure at the source, before other forces change its nature. For this we will initially need a large vibrating area with a low fundamental frequency, such as the tympani.

THE DIAMETRIC NODES

Of one thing we can be sure. The function of the air when it behaves as a carrier and the function of the air when it behaves as a generator are quite different in nature from one another. What makes any gas unique is its compressibility. The carrier wave is, in fact, made up of spherical volumes of air, which are compressed followed by the reciprocal version in which the air is rarified.

We define this carrier structure as the 'sound wave' (it is, after all, what actually strikes the ear drum). We seem to be unaware of the degree of modification this carrier structure imposes upon the basic generator structure. The carrier wave always behaves sympathetically. It cannot generate sound. To look for the generating structure in the carrier wave is a mistake.

Yet this is invariably what we do. What we see is a structure that seems to support the notion that a sound is made up of a fundamental plus its overtones. What we are analyzing is the airborne carrier wave. We assume that the information carried by the pressure wave is the same as that of the body that creates it. We overlook that fact that air, being a gas, has properties that affect the initial structure to a significant degree.

We must go to the source and look for the makeup of the structure as it appears on the instrument itself. We will examine this first on the membrane of the tympani, mainly because the tympani lends itself so well to a fairly simple set of experiments.

There are two generally held assumptions regarding the tympani. First, the membrane vibrates by sounding first the fundamental and then the overtones in sequential fashion.

Second, the tone one hears is thought to be the fundamental. Most analytical thinking starts here.

If, indeed, a vibrating body sounds every overtone it must have the inherent potential to sound every frequency in the series and this should be easily detectable. If the membrane of the tympani is actually generating each and every overtone of the structure as we read it from the airborne carrier, then the membrane must have the capability of responding to the frequency of any given overtone. Experiment, however, proves that this is not true.

If the membrane of the tympani is struck it emits a sound that is definitely musical. This is a subjective definition. By calling it musical it is meant that it is possible to discern its pitch (this paper puts forth the notion that all sound is pitched and that noise is subjective, just as is the notion of dissonance). The tympani, unlike the bass drum, the snare drum, the tom-tom, the conga drum, etc., sounds an easily discerned pitch. Here is a clear example of how we are trapped by our senses. All sound is the result of an aggregate structure formed by the differential result of an input structure, which is integral.

Since it is impossible to sound a complex of simultaneous frequencies that are not integral it is impossible to sound a complex of frequencies that will not produce a fundamental. All sound is pitched. The only difference between the tympani and the other percussion instruments mentioned is that the pitch of the tympani is discernable. The fundamental of indiscernible pitched sound is remote and often is the result of many pitches sounding together and this is why we cannot discern it. The fact that we cannot hear the pitches of various drums does not make them any less real.

If there is a basic input structure that will in turn differentially create the sequential structure our analyzers tend to see, then it must, in fact, be the only sounds the tympani is capable of producing. Unfortunately when we hit the drum we get everything. It is very hard to break the patterns down into the basic forms.

Because of the size of the membrane we can affix a small piece of ferritic metal to the drum without seriously affecting the structural patterns that develop. Then, by use of an electromagnet, we can inject any frequency we choose.

If we sprinkle the membrane with a fine black powder we have an extremely sensitive detector of any minute change in the topological configuration of the drumhead. The black powder instantly displays any change in the drumhead caused by the input frequency.

If we use, as our input, a variable oscillator and begin well below the pitch of the tympani, then gradually raise the input frequency, we find that through most of the range of input frequencies nothing significant happens to the powder on the membrane. At several distinct pitches, however, the powder very dramatically forms itself into precise lines. These lines, in turn, form simple, basic geometric patterns, the Chladni patterns (named for their discoverer). The same pattern always appears at the same frequency. In the case of the tympani the pattern will be one of three types, or modes.

One type is formed from concentric circles. These always divide the diameter into sections with isometric diameters. The second is formed from intersecting diameters, dividing the membrane into isometric, pie-shaped segments. The third type is a hybrid, formed of a composite of the circular and diametric modal types.

Because all three types of patterns appear on the face of the tympani we assume that they all represent the same function. A closer look at the nature of the tympani would seem to indicate that the various pattern types, while related, are the results of different functions of the vibratory pattern of the drumhead. It would also seem that we have a somewhat incomplete notion of what function of a vibrating body creates the sound, especially musical sound.

In all likelihood the only patterns we will need concern ourselves with are those that are formed by intersecting diameters. The other two most likely represent different vibratory functions. In a freely vibrating body we may find that there are possibly no more than six of these diametric nodes.

OSCILLATION

The two most significant elements in any musical instrument are materials from which it is made and the shape. The mathematics developed thus far support a notion that acoustical phenomena are much more inherently consistent than we have imagined.

One of the most constant factors in musical instruments is that at least one cross-section of the oscillating body is usually circular. All strings have circular cross sections. Virtually all wind and brass instruments have circular cross sections. The rare exceptions, such as the ocarina, are quasi-circular. In the few exceptions, such as xylophones, glockenspiels and cowbells the circle is replaced with a rectangle. This consistency suggests that the carrier wave is always exited in the same manner. This is not the way in which it is viewed in contemporary thinking.

The membrane of the tympani is stretched over a frame that is round. The membrane itself is a cylinder (we need only consider that part of the membrane within the steel ring. That is the part that is free to vibrate). The body of the tympani is a modified cylinder. This modification is also consistent. Just as with virtually all other instruments, tympanis are all shaped exactly alike.

All sound is initiated by something that oscillates. The uniform, reciprocal shape of all musical instruments directs most of the available energy into a few very low overtones, or possibly more correctly, modes of oscillation. Irregularly shaped vibrating bodies allow the bulk of the energy to remain in very complex structures that lie high in the series. However, the structures of both uniform and more randomly shaped vibrating bodies differentially create a fundamental. Noise as a physical reality does not exist. All sound must be created in precisely the same fashion.

Let us return to the Chladni patterns. If we strike a tympani exactly in the center of the drumhead the result is a dull, unmusical thud. This is again consistent, in no musical instrument is the energy ever injected into the center of the oscillating member. If we feed the output from our variable oscillator into the center of the drumhead only the circular nodes appear.

In order to create the lovely characteristic sound of the tympani one must strike the drumhead some distance from the center. Different distances produce different timbres. This is true for all drums. When the variable oscillator is fed to a point away from the center the diametric and hybrid nodes appear as well. The circular nodes themselves are unaffected by the off-center energy input. This forces us to assume that the circular nodes may not be part of the process that creates the actual interface to the airborne carrier.

There are several logical arguments to support this assumption. Strings and drumheads have one significant thing in common; they are both cylinders with parallel sides. The major difference is in the ratios of the lengths to the diameters. In the string the length is enormous in comparison with the diameter and conversely, in the drumhead the diameter is enormous in comparison with the length.

A cylinder whose length is the same as its diameter would oscillate poorly no mater in which direction the energy input should be. By making one dimension very small and the other very large we make one direction a more productive oscillator than the other. In the string this motion is parallel to the plain of the diametric cross section and in the tympani it is at right angles to the plain of the diametric cross section. In both strings and tympanis the oscillating membranes are placed under tension. In the string this tension is in the direction of the length and in the drumhead is consistent around the edge of the drumhead.

By actual definition neither oscillation, or for that matter, vibration adequately describe what actually happens to the strings or membranes when they are hit, scraped or plucked. If either the string or the drumhead merely moved to and fro, as would a pendulum, no significant structure would be formed and thus, no musical sound.

What a vibrating, or oscillating membrane does is much more complex than just moving back and forth. Pendulums oscillate; strings and drumheads do something else entirely. Once we have defined its nature, the oscillating phase of sound generation will concern us little, thus it should be sufficient to use the term oscillation with the understanding that we mean the complex of motions that comprise the vibration.

OSCILLATORY MOTION

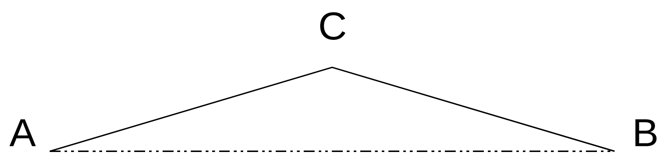
Consider the nature of a string under tension, and that this tension lies between two terminal points, A and B.

A ______________________________ B

It is impossible to pull this string from one end without the other end producing an effect exactly like the end that is pulled. No matter from whence the energy comes the string reacts as if the pull is equal from each end. This gives rise to some interesting thoughts, which will be pursued, in later sections Physics tells us that the force when the string is pulled is equal along the length of the line. However I am treating this as if the force were to be greater in the center of the line. This is because of the relationship of the string and the drumhead. IT is an interesting speculation in any case.

As a string is pulled the mass of the string has to occupy a greater length. Since the mass is constant the only reaction must be for the string to reduce its diameter. Lord Raleigh's restoring force causes the string to attempt to regain its original diameter. This in turn is what creates the pull on both ends. The buildup of forces is greatest in the center of the string. Anyone who has ever heated a glass tube in a Bunsen burner and then pulled the ends apart has seen this effect.

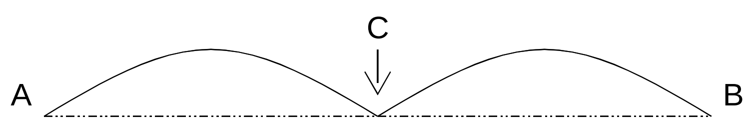
Let us assume a force pulling the center of the string in a direction at right angles to the direction of the standing force. Such an application of force would produce this:



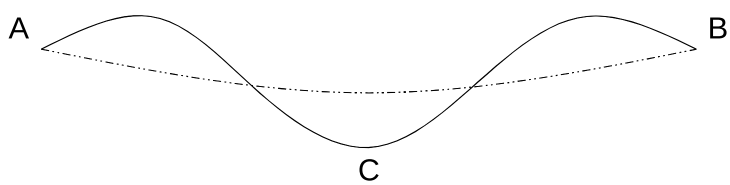
This increases the force at C, which already is affected by the greatest buildup of forces. The shape the string is now oblong and not only is there a force pulling on the end but as the string attempts to regain its shape it now is affected by the transverse force of the oblong shape. The tendency is for the string to vibrate until it returns to the rest position. If the string is well constructed this can take considerable time. Observation tells us that this process is not simple. It does not just return to the center and then swing out to the other side. That is what would happen if it were simple harmonic motion. If this was what happened this oscillation would not generate the standing structure that produces timbre.

Let us speculate on the action of the string when the pull on point C is released. Point C has the largest buildup of force. If this were not true we would expect the string to merely return to the starting position, then swing out the other way, as mentioned before. At the risk of oversimplification we can assume that the string must behave to some degree in the following manner.

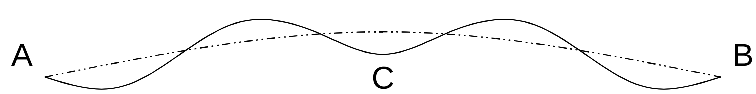
The released point C will move much faster than the points of the string closer to the ends. The only movement possible is exactly perpendicular to the original line AB. Point C should reach the starting line first and at the end of the first half period the string should look like this.



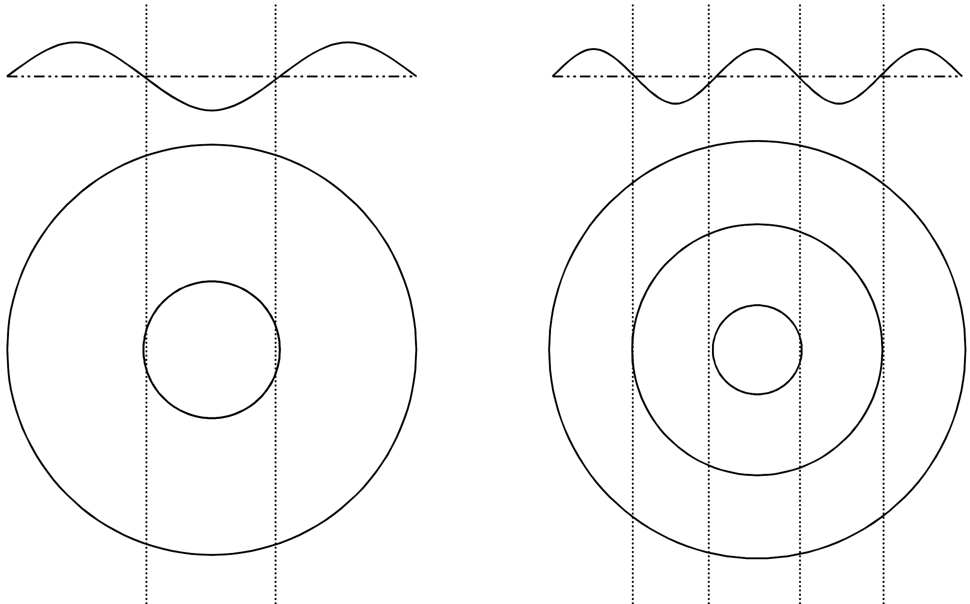
Point C does not stop as it reaches the starting line and so it keeps moving. Once it passes the starting line the reciprocal of the original forces occur and the structure should become something like this:



Things now get very complex as, while point C still is the point where the buildup of forces is greatest, we now have parts of the string moving in opposite directions. What happens next? My own feeling is that the next swing back to the top half of the cycle would produce five arches:



I have a reason for suggesting this. Consider these consecutive, basic shapes on both the string and the drumhead.



The lines formed by the string in the previous example are exactly what any diameter of the drumhead would look like if viewed from the edge of the membrane. This gives us cause to assume that the circular nodes make up the oscillating phase of the structure. This division creates equal volumes in the string. The same equal divisions in the tympani create volumes that are in octave ratios.

In the first figure, if the diameter of the inner circle is **1** and the diameter of the outer circle is **3** we can observe that the volume of the inner circle is $1^2 * \pi * h$ and the volume of the outer ring is $3^2 * \pi * h - 1^2 * \pi * h$. Since in this case π and h are constant the ratio of the volumes is $8/1$ or three octaves. Thus in the second circle, if the diameter of the inner circle is **1,** the volume of the outer ring is $5^2 - 3^2$ or **16,** four octaves lower than the inner circle and one octave lower than the middle circle.

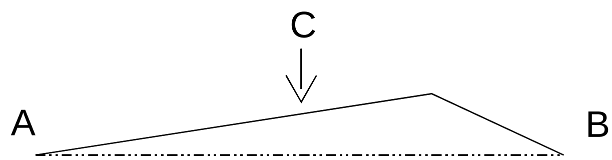
If a timpani is struck in the very center of the drum the sound is a dull thud. The same thing occurs if a sting is plucked in the very center. Vibrometer studies of the surface of a timpani show this very clearly.

We can see empirically that the oscillating phase does not produce the sound. A vibrating string that is not directly connected to a resonator of sufficient size can scarcely be heard, and the sound that is actually heard is quite fuzzy and diffuse with scarcely a hint of timbre.

Such movements would be difficult to detect with a strobe as the shape of the string changes every half cycle. This would create a rather diffuse, fuzzy image. Even if the strobe were to be in phase with the vibrating string the results would be blurred. I suggest that observers attribute this fuzziness to super positioning that is; the string is thought to be sounding all of its partials together. If this mode of vibration holds true then it must be true for all vibration

The mouthpieces of brasses as well as the reeds of woodwinds, if removed from the main body of the horn also sound a fuzzy diffuse sound, sort of like a Bronx cheer. The oscillating phase simply does not create the standing wave that ultimately gives us the sensation of timbre.

The fact that the diametric nodes (and thus the timbre) only appear when the energy input is off-center indicates still another motion in the drumhead. Let us pull our string back in a point that is not the center creating this:



The point of greatest build up of energy still remains at point C and thus when the string is released, point C gets to the starting position first as with the centered input. There exist a lag, however between right side of the structure and the left, creating a sloshing motion that ripples back and forth like waves in a bathtub. It is this undulation that creates the diametric nodes that, in turn, create timbre. This is just as true in the string as it is in the drumhead.

The interesting thing here is that the motion behaves in much the same manner with or without the undulation. Any given point on the string can only move perpendicular to the length. The longitudinal shock waves formed by the flexing of the string are the same whether the energy is injected in the center or not. This apparent lateral motion must therefore cause the diametric nodes. This motion behaves much like a wave that forms in water in that the only actual motion is up and down but the apparent wave is longitudinal. The distance from the center is critical to the sound in any instrument.

There is not one musical instrument with a string or membrane under tension in which the energy is introduced into the center of the oscillator. All such instruments inject the energy some considerable distance from the center. This distance is reasonably constant among the various instruments and the math is reminiscent of that which controls the length to diameter ratio of a rocket. The cylinder is, is all cases, a musical shape, no matter to which use it might be put.

THE INTERFACE

Before we return to the diametric nodes we must first take a look at the manner in which the resonating structure presents its information to the airborne carrier wave. Here, as with most other musical phenomena we find absolute consistency. The primary point of interface to the carrier wave is always across the plain of the circular cross section. This is true even if the resonating body is air. We shall discover that it is the standing structure that forms in the air that is the primary interface in the majority of all musical sounds, and quite likely, all sound.

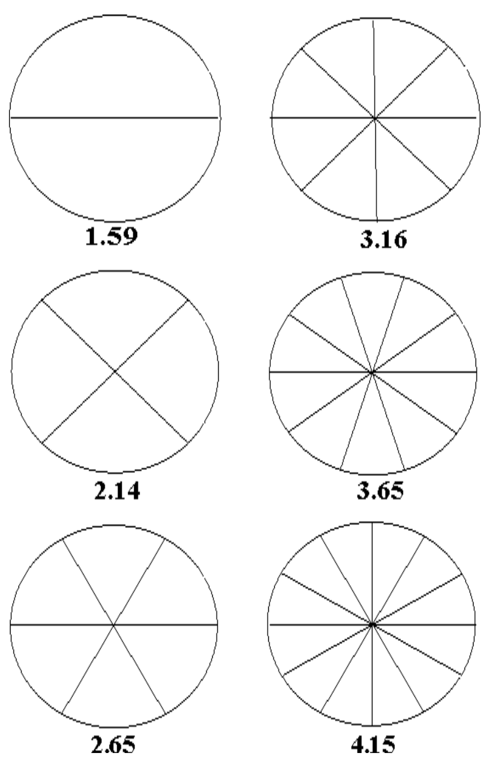
That is not to say that some sound is radiated from other sources and in other directions. A certain degree of the sound emanates from the sides of a trumpet but the main body of sound emanates from the interface that in this case is across the plain of the end of the bell. There is a striking orchestral device wherein the French horns are told to play a given passage 'bells up'. This points the bells right at the audience. This makes the sound of the horns appear to the listener as being much louder. The sound that emanates from anything other than the cross section of the diametric nodal structure is in sympathetic response with the diametric structure. We can state with sufficient accuracy that the thrust of the sound is in the direction of the length of the cylinder and perpendicular to the cross section of the diameters. The fact that there is certain spill to the sides of an instrument could prove to be very useful in analyzing these patterns. Since any resonator will modify the sound the sound from the sides of any instrument will color the sound to a degree. That will present no problem as it can be mixed with the primary structure. The ear receives simultaneous sounds as one structure so a mix of sounds will cause the ear to respond exactly the same as when excited by individual sources. Another example is a cathedral bell. When the bell swings toward the listener the sound is quite loud. When the bells swings away the sound is reduced considerably.

The tympani is no different. The major input of sound into the carrier is on the surface of the drumhead. If we remove the drumhead and its tension creating devices from the 'kettle' of the drum we discover that the drumhead, when struck, produces a small, short lived, diffuse and nearly timberless sound. This leads us to believe that the drumhead does not create the diametric structure but, in all probabilities, sympathizes with it. It is reasonable to assume that the standing structure forms in the airspace of the tympani.

Let us return now to the Chladni patterns. We can now support the notion that the diametric structure is all that we need seek, and we can do this, mathematically.

THE DIAMETRIC STRUCTURE

One of the things that has baffled acoustical physics for a long time is that the nodal structure of the tympani appears to be composed of partials that are not harmonic. Let us consider the first six Chladni patterns developed by intersecting diameters. The numbers beneath each figure are the frequency ratios of these particular nodes in an ideal drumhead.



The lines represent the part of the drumhead that is not moving. These lines, in each case, divide the drumhead into pie-shaped segments of equal volume.

Since the number of segments of any number of diametric divisions will always be equal, it follows that any opposing segments will be moving in the same direction and adjacent section will move in opposite directions.

The numbers under the diagrams have been a mystery for years, as they appear to be non-harmonic whereas all tympani create a sound that is discernibly pitched and rather pleasant sounding. This would cause one to believe that the nodal structure would be a simple structure composed of familiar musical intervals. The common explanation is that the tympani 'works' because the nodes approximate the basic harmonic series, and indeed they do, but in a slightly different manner than is currently accepted.

If we look at the first two nodes intervallicly we see that

$$2.14/1.59 = 1.345911949685$$

This is very close to a perfect fourth of **1.33333**, missing by only **1.00943396**. This odd interval is the result of a prime comma, $107/106 = 1.00943396$. It is very close to being a perfect fourth. The ratio seems to be meaningless so we shall for the moment take a slightly different tack. **1.59** is very close intervallicly to the common musical ratio of **1.6**, the difference being **1.00628931**.

This is also a prime comma,

$$160/159 \ = 1.00628931.$$

This is a more interesting comma as **160** is the fifth octave of **5** (**5 x 2^5**). If we let **1.6** replace **1.59** in the ideal series the first interval is now

$$2.14/1.6 = 1.3375$$

This misses **1.333333** by **1.003125**. This is the result of a prime interval

$$321/320 = 1.003125.$$

320 is the sixth octave of **5 (5 x 2^6**). If we assume the actual ideal to be $4/3$ then we should multiply **1.6** by **1.333333** giving us a value for the second node of **2.13333333**. If we divide this by **2** and bring it into the prime octave we have **1.06666667**, another familiar musical ratio.

$$2.14/2 = 1.07$$

$$1.07/1.6666 = 1.003125\,,$$

These ratios, as before are the result of the prime comma, $321/320$. Half of **2.65** is **1.325** very close to **1.333333**, missing by

$$1.33333/1.325 = 1.0062892830$$

again, the same prime comma as the first, $160/159$.

Half of **3.16** is **1.58**. This again is very close to **1.6**, missing by **1.01265823** or the prime comma

$$80/79 = 1.0126582278481012658.$$

Half of **3.65 = 1.825**, missing **1.8** by **1.01388889**, produced by the prime comma,

$$73/72 = 1.013888888889,$$

this latter being the first ratio that does not contain, as one of its members, an octave of **5**. In this instance the ratio contains an octave of **9** **(9 x** 2^3**).**

4.15 must be divided by four as it jumps to the third octave. $4.14/4 = 1.0375$. If we then let **1.0375** be equal to **1** we obtain the following chart:

Diameters	I	II	III	IV	V	VI	VII	VIII
(6)	4.15	1.0375	1.0	4.0	$83/80$	above	30	15
(5)	3.65	1.825	1.8	3.6	$73/72$	above	27	27
(4)	3.16	1.58	1.6	3.2	$80/79$	below	24	3
(3)	2.65	1.325	1.3333	2.6666	$160/159$	below	20	5
(2)	2.14	1.07	1.0666	2.1333	$321/320$	above	16	1
(1)	1.59	1.59	1.6	1.6	$160/159$	below	12	3

Column I lists the ratios of the ideal drumhead as given. Column II lists the same as they appear in the prime octave. Column III lists the common musical ratio that is closest to each. Column IV lists column III returned to the same octave as column I. Column V lists the ratios of the differences between either I and IV or II and III. In the first four nodal structures the even member of the ratio is an octave of **5**. (**80 = 5 x** 2^4, **160 = 5 x** 2^5 and **320 = 5 x** 2^6). The even member of the remaining structure (**5** diameters) is an octave of **9** **(9 x** 2^3**).**

Column VI indicates whether the value in column I is above or below the corresponding value in column IV. What it is that causes these small but extremely precise commas is hard to imagine. Lord Raleigh's assertion that this is caused by stiffness of the membrane does not seem to be correct as sometimes the sounding node is above the natural value and sometimes it is below. The problems in measurement that are usually experienced in these matters are caused by not assuming the principal tone (the tone one 'hears') to be **3**. The principal tone is equal to the nodes of one diameter or four diameters. The fundamental is equal to the node of two diameters.

Column VII is column IV represented by integers. This is arrived at by multiplying column IV by **7.5**. Column VIII is column VII reduced to initial overtone entry values (odd numbers).

If we divide the first four values of VII (**12, 16, 20, 24**) by **4** we get the following:

24	**6**	**(3)**
20	**5**	**(5)**
16	**4**	**(1)**
12	**3**	**(3)**

This center column is the familiar six-four chord. This time it appears with a doubled fifth **(3** and **6**), which is just the way it would be normally used in actual composition. Consider just how basic the triadic structure is in the enharmonic matrix.

3	
1	**5**

If we examine the entire set of six diametric structures as they appear in the matrix we get:

27	**x**
x	**x**
3	**15**
1	**5**

This certainly lies within the confines of the scale structure composed of musical reciprocals of **135**. It also comes very close to being the pentatonic scale. This offers the first support to my idea that the pentatonic scale is so ubiquitous because of the fact that we hear it, to one degree or another, in every sound we hear.

Recall the hierarchy of musical ratios as listed at the end of part 1.

81/80	**1.0125**
25/24	**1.04166666**
16/15	**1.06666666**
10/9	**1.11111111**
9/8	**1.125**
6/5	**1.2**
5/4	**1.25**
4/3	**1.33333333**
3/2	**1.5**
2/1	**2.0**

If we list column VII as intervals we achieve:

30/27	**1.11111**
27/24	**1.125**
24/20	**1.2**
20/16	**1.25**
16/12	**1.33333**

If we consider those very tiny comma displacements as being substitutes for the simpler interval such a complete hierarchical structure supports the notion that the diametric nodes, in themselves, create the sound. Should this be true then the actual nodes of the drumhead replace the above structure.

Recall again the initial chart:

Diameters	I	II	III	IV	V	VI	VII	VIII
(6)	4.15	1.0375	1.0	4.0	$^{83}/_{80}$	above	30	15
(5)	3.65	1.825	1.8	3.6	$^{73}/_{72}$	above	27	27
(4)	3.16	1.58	1.6	3.2	$^{80}/_{79}$	below	24	3
(3)	2.65	1.325	1.3333	2.6666	$^{160}/_{159}$	below	20	5
(2)	2.14	1.07	1.0666	2.1333	$^{321}/_{320}$	above	16	1
(1)	1.59	1.59	1.6	1.6	$^{160}/_{159}$	below	12	3

There is a mathematical discrepancy between the first four columns and the last two. Columns I and IV would suggest that the node formed by six diameters is the fundamental (**15** in column VII). On the other hand columns VII and VIII indicate the node created by two diameters is the fundamental. Neither is the principal tone. If we bring column VIII into the prime octave we see that each value in column IV is higher than its corresponding value in column VIII by **1.06666**.

$^{1.6}/_{1.0666} = 1.5$, and this is more practical. It is usually assumed that the first actual node is one in which there are no diameters. I do not think that this is true. In every other nodal structure we have two equal masses ‘moving' in opposite directions. This is not true if there are no diameters. The fundamental is a result of, not a participant in the structure, and thus is created differentially.

If we divide everything in the first four columns by **1.0666** we get the following:

Diameters	I	II	III	IV	V	VI	VII	VIII
(6)	3.890625	1.9453125	1.875	3.75	$^{83}/_{80}$	above	30	15
(5)	3.421875	1.7109375	1.6875	3.375	$^{73}/_{72}$	above	27	27
(4)	2.9625	1.48125	1.5	3.0	$^{80}/_{79}$	below	24	3
(3)	2.484375	1.2421875	1.25	2.5	$^{160}/_{159}$	below	20	5
(2)	2.00625	1.003125	1.0	2.0	$^{321}/_{320}$	above	16	1
(1)	1.490625	1.490625	1.5	1.5	$^{160}/_{159}$	below	12	3

This is more practical representation. The fundamental will be created differentially so we can consider VII and VIII to be the structure of the tympani. To illustrate just how small these displacements are here is a chart of the actual as opposed to the ideal overtones of a tympani sounding a low **F.**

The tone that is heard, **80Hz**, is the third overtone thus the fundamental is **53.3333333**.

Matrix number	Actual Pitch	Ideal Pitch
15	207.5	200
27	182.5	180
3	158	160
5	132.5	133.333333337
1	107	106.666666667
3	79.5	80
(1)	53.3333333	53.3333333

Another way of looking at this is to start with the audible of principle tone. In this case the actual fundamental would be **53.6687631** and the chart would be thus;

Matrix number	Actual Pitch	Ideal Pitch
15	208.805031444904	200
27	183.647798740699	180
3	158.993710690578	160
5	133.333333332289	133.333333337
1	107.672955974	106.666666667
3	80	80
(1)	53.6687631	53.3333333

In either case the pitch differentials are slight, especially in the lower partials. Experiment is necessary to see if the ideal pitches produce the same basic sound as the actual pitches. The differences are small enough that is quite probable that the sounds will be basically the same. The actual pitches should produce the ideal sound, something that may be very desirable.

The same type of patterns should appear in all other sounds. Another set of patterns shows this to be the case.

CONSISTENCY IN THE DIAMETRIC STRUCTURES

As a musical tone deteriorates the upper partials go first. They require and expend more energy. We can assume that the bulk of the sound will be maintained by some form or other of the six-four chord

In a test devised some years ago by Frederick Saunders, Carleen Hutchins and Alvin Hopping, brass plates were cut in the shape of the body of a violin. In the violin, the top and bottom of the sound box behave, collectively, like the drumhead of a tympani. These brass plates allowed them to test the Chladni patterns of the violin just as with the drumhead.

As with the tympani, six nodes were listed.

(6)	**1600**
(5)	**780**
(4)	**520**
(3)	**435**
(2)	**340**
(1)	**260**

As with the tympani the observed values miss being simple musical values by small, prime commas.

	I	II	II	IV	V	VI	VII
(6)	**1600**	**6.15384615**	**6**	**1560**	**above**	**40/39**	**8**
(5)	**780**	**3**	**3**	**780**	**x**	**x**	**9**
(4)	**520**	**2**	**2**	**520**	**x**	**x**	**6**
(3)	**435**	**1.67307692**	**1.66666**	**433.33**	**above**	**261/260**	**5**
(2)	**340**	**1.30769231**	**1.333332**	**346.66**	**below**	**52/51**	**4**
(1)	**260**	**1.0**	**1.0**	**260**	**x**	**x**	**3**

Again, the values in column I are frequencies given by the authors. Since **260** is the principal tone **(3)** let us divide column I by **260** creating column II. As before column III lists very close basic music ratios and represents column III multiplied by **260**. Column V lists the direction of the discrepancy. Column VI lists the prime comma involved in each discrepancy. Column VII is column III multiplied by **3**, giving us the smallest series of integers that are possible (unclouded by commas). Once again we see that the first four notes are the six-four chord.

The two remaining nodes are both octaves of nine. In the matrix the structure it would be thus:

9	
3	
1	**5**

As with the tympani the principle tone is **3**. Since the differences between the actual and ideal pitches are formed by prime commas it may be that the ideal tones will work. If this proves not to be true it will be no problem to alter the ideal tone so as to produce the actual. This will be a main goal of initial experiments.

Again it leans toward the pentatonic. Can a tone as rich as that of a violin be formed from such a simple basic structure? The previous example uses a metal sheet in the shape of a violin so tests on an actual violin will be needed but we can speculate that the some form of this simple structure would indeed create the sound of a violin.

Remember that apparent sounds of both the violin and the tympani are made up of a very large complex overtone pattern. We can expect that the structures that create the sound of most instruments will be composed of structures no more complex than those of the tympani and the violin.

Laser holography allows us to produce the 'Chladni patterns' of any instrument or, for that matter, any object.

It was stated in Part I that it is not the air rushing in and out of the finger holes that produces the airborne carrier wave, that which we usually refer to as sound. It is not the movement of the string or the blade of a tuning fork nor the sloshing of the air back and forth in a wind instrument that creates the sound. This oscillation must create another structure that actually is the interface to the air-born carrier.

The length of the bore itself in brasses controls the pitch. Let us differentiate these by correctly calling the movement oscillation and the structure that the movement creates, resonance. There can be no doubt that this resonant structure runs the entire length of woodwinds as well, regardless of the length of the oscillation (the addition of a bell affects all of the notes). All sound is created by structures that are very similar. It would appear that all musical sounds are formed upon the basic six-four-chord structure; most likely all sound is similarly structured.

A certain amount of sound is transmitted in directions other than perpendicular to the circular plane. We can assume this is a sympathetic response to the diametric structure itself. These structures can be very powerful.

The diametric structure runs the length of the bore of a wind or brass instrument and the body of the instrument itself responds to this structure.

We can now go back to the tympani and draw a few conclusions,

DIAMETRIC RESONANCE

At first glance it would appear that the diametric structure on the tympani is formed in the drumhead, as are, no doubt, the circular nodes. The elastic force being applied (which is considerable) is in a direction that is directly opposite to the direction of the diametric nodes.

We must always be aware that while the techniques used in eliciting the Chladni patterns do indeed show us the basic nodal structure of the vibrating body, this process is considerably different than what happens in a naturally sounding tympani. It likely that things happen in one mode that would not happen in the other. The circular modes in a natural sounding drum are caused by the oscillation of the drumhead and, as with all oscillations they contribute nothing significant to the sound. All drums are rather unique as the drumheads behave as both the vibrating body and the resonant structure that actually interfaces with the air.

On the other hand, the fact that an arithmetic, or linear progression of equal divisions of the mass of the vibrator produces a series of non-linear frequencies that is common to both the induced and natural modes. The frequency ratios are consistent as well.

The type of shock wave created by the undulations of the drumhead would move in this opposite direction. What these shock waves actually are is of no importance to this discussion, thus shock wave is as good a term as any. It is quite likely that the oscillation of the drumhead has little effect on the either the airborne carrier or the air within the drum itself.

The design and construction of the tympani is, as previously mentioned, remarkably consistent. This would lead us to believe that the sides of the drum are much more involved with ensuing processes than merely to provide the necessary shape. If shape were all that mattered the drum could be made of Plexiglas, and this is not true.

The metallic structure that we might call the 'kettle' of the drum must act in much the same way as the steel frame of the piano or the body of a brass instrument, that is, as a conduit to the resonator. This makes the resonator, that which creates the airborne carrier, the air-space within the drum. It is, in fact, a rather unique form of cylinder. The head of the tympani serves as both ends of this cylinder. The shape of the kettle is designed to throw back a conjugate of the input of the drumhead. Its particular shape allows it to accept the tuning a thus the diametric structure of any fundamental frequency.

We can deduce from observation that the resonating structure that forms in an air filled cylinder is much more powerful than the same happens in a solid. The area of the cross section across the end of the bell of a clarinet is much smaller than the area of the tympani or the sounding board of a piano. It would appear that the area of the interface has as much, and no doubt more, to do with the loudness of the sound than the initial available energy.

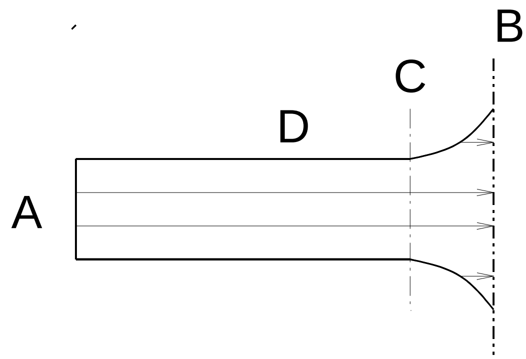
The logical place for this diametric structure to form (remember that the diametric structure is always three dimensional) is in the air filled cylinder. The tympani is actually a muted wind instrument. All drums are. The patterns we see on-the drumhead are created, for the most part, sympathetically, i.e., it does what the air within it does.

It would be important to remove the kettle from the ring stretching the drumhead and, while maintaining the tension, again induce vibration into the drumhead magnetically. If the diametric nodes do not appear we can be reasonably sure that the diametric structure forms in the air chamber. If they do appear at all it may be that they are created by other things, such as the ring itself. Other experiments will confirm the reality of the structure formed within the tympani.

Considering the awesome amount of energy held in a piano string it is odd that without the heavy body of the piano as sympathetic resonator the vibrating string can scarcely be heard at all. The tension in a tuning fork is inherent in its shape and the tuning fork suffers the same fate as the string, it can scarcely be heard without a resonator.

If we assume that all sound is formed by the diametric structure, which is always longitudinal in nature, this seeming paradox between available energy and the sound we hear is easily explainable. The areas of interface that feed the energy into the airborne structure in either the string or the tuning fork is extremely small. As we have seen solids such as steel are much less efficient in the structural transfer to the airborne carrier than structures that form in the air. This makes the area of interface of strings and tuning forks react very poorly indeed.

Even in some woodwinds and all brass instruments the bore that allows the oscillation is too small to produce the sound we desire. In this case a bell is added. If thought of as an extension of the long diametric structure a very simple explanation offers itself. Because of the nature of the diametric structures these structure must adhere to the principals of hydrodynamics.



The drawing is a longitudinal cross section of a cylindrical bore with a bell, such as exists on a brass instrument. Let us assume that the volume of air with the instrument itself is replaced with a fluid. If a force is applied at A in the direction of the arrow we know that the force per square inch of a transverse cross section at C is equal to the force per square inch of a transverse cross section at A. The total amount of energy is proportionate to the ratios of various cross sections, i. e., if the area of a transverse cross section at B is four times that of one at A or C, the total amount of the force is four times as great.

Considering the chamber once again filled with air let us assume a finger-hole at D. We know from observation that the oscillating structure will form between A and D. The volume of air from D to C remains relatively undisturbed and yet the interface to the carrier is a B. The structure found across a transverse cross section of B must be a continuation of a structure that exists within the cylindrical bore.

We can only assume then that the oscillatory phase of this phenomenon creates a totally different structure that runs the length of the bore regardless of the length of the oscillation. It would also be logical to assume that this structure is diametric since the direction of the energy is longitudinal. This diametric structure is governed by the principles of hydrodynamics, i. e., in a resonating structure a gas behaves as if it is a fluid.

The air created resonance is more common that one would suppose. It is the air space in violins, violas, cellos, guitars and all similar instruments that is the primary interface with the airborne carrier. The F-holes are not cut into the precisely fashioned sounding boards for decoration. The cumulative area of the F-holes in a violin is at least as much as is that of a woodwind. Even a loudspeaker feeds into an air filled box. Without this air filled space the loudspeaker is much less efficient. The diaphragm of the speaker is invariably made in the shape of a cone and such the structure that interfaces with the airborne carrier still forms in an air filled chamber.

While the string develops diametric structures longitudinally just as does any vibrating body, the process of transferring the structure to the resonator is different. The only points the string touches are the tuning pins on a piano. On a sting instrument this primary contact is made between the edge of the string and the bridge. No contact is made through the powerful transverse cross section but rather on the side. This would seem to be a very inefficient method of transferring the structure and indeed, only about two per cent of the energy injected into the string of a violin actually creates the diametric structure. The same is true for a tuning fork. When the fork is placed on a resonator only one point of the end (usually rounded) of the fork comes in contact with the resonator.

It is usually assumed that the lost ninety-eight percent of this energy is turned into heat. The energy injected by a single stroke of the bow would be relatively small. The cumulative energy from an entire complex piece, such as a Paganini Caprice would be quite considerable. Since wood conducts heat very poorly we would expect that even in one such piece something in the violin would get fairly warm. This does not happen.

It is more likely most of this ninety eight per cent of the energy is never transferred as the transfer is made through the side. No doubt some of the energy is turned into heat but most likely it, as with the energy that creates the sound, is small. It is this ability of the diametric structure to be detectable through the side that may prove very useful in analyzing these diametric structures.

THE SEQUENTIAL STRUCTURE

This concept of structural energy being 'spilled' from a diametric resonance is not all that alien to that which is already accepted. Complex equations exist to calculate the amount of energy that spills from a naturally sounding horn, for instance. What the diametric nodes of the tympani show is that there are quite likely no more than six precise, discrete frequencies that the tympani can sound. Whatever energy it emits it does at one of these frequencies. The impulse structure that initiates everything may be more complex than that of the structure that comes from the device that strikes it. The size of the striker for any such instrument is critical to the quality of the sound. This is true for drums, xylophones, glockenspiels and cathedral bells. If the mass of the striker does not form a simple ratio with the mass of the target the amplitude and quality of the sound is diminished.

This choice of available frequencies is further reduced by the fact that the higher nodes disappear rather quickly. Thus the bulk of the duration of any sound (even a sound so short as a tympani) is made up primarily of the first four nodes, which, most likely will be a six-four chord. Musical sounds must be composed largely of these four nodes. The question is how such a simple structure as a six-four chord could create the multiplicity of tone colors we experience.

This same six-four chord comprises the notes of a bugle, and from the bugle we get a clue to physical reality of the nodal structure of the tympani. The most salient feature of the bugle is that one, and only one, sound can be sounded at any given time. Simultaneous sounds are a physical impossibility. The same must be true of nodal structures. Only one diametric structure can form at a time. Thus the input that creates the sound wave is not only restricted to a few frequencies but it is sequential as well. This explains many things.

Now it becomes easier to see how so few inputs can create such a variety. The number of tunes one can create, even with as few as four inputs, is vast indeed. In a later section we shall see how this sequencing explains the ability to transmit amplitude in a non-compressible medium such as steel or water.

This raises a serious question. Harmonic analysis shows more, and often many more, sounding overtones than those we would expect from the diametric patterns. We can expect overtones higher than six even though the entry structure may not exceed that number. Additive tones have been mentioned in many theoretical writings and that is probably what causes the higher partials to form. The fact remains, however, that the diametric nodes indicate a very restricted number of possible frequencies the drum can sound. Even if we were to add the circular and hybrid nodes the number remains very small. The drum cannot create its overtone structure by arithmetic progression of all of the overtones, as it is incapable of producing most of those frequencies.

Here we need a special kind of testing and the tympani would seem to be a good place to begin. The face of the drum presents a large area and the pitches involved are relatively low. We must observe the topology of the drumhead as it interfaces with the carrier. Several possibilities exist and this will be the thrust of the initial experiments. We must perform the experiments as if the airborne carrier does not even exist.

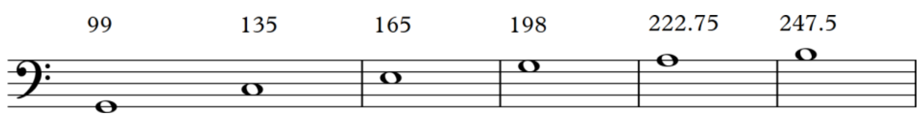
We can assume, since the matrix structure includes **27**, that the implied structure is the two-column structure generated by **45**.

27	**x**
x	**x**
3	**15**
1	**5**

The implied structure is **135** and experiments that incorporate that value would be desirable.

We must sample the topography of the drumhead exactly **135** times the fundamental. This value must be high enough to allow for the fact that the series does not appear in the basic matrix form, **3, 1, 5, 3, 27, 15**, but rather in scale form, **12, 16, 20, 24, 27, 30**.

Let us assume a low **G** of **99Hz.** The actual series would then be:



We should discover a basic pattern that varies from the beginning to the end of the actual sound. It is this sequence of diametric frequencies that is responsible for the phenomenon we call tone color, or timbre.

Each instrument will have its own distinctive diametric sequences. Poor sounding and beautiful sounding versions of the same instruments should produce detectable variations in a common basic pattern. This basic pattern will point to an ideal structure that is impossible to achieve with acoustical instruments. Electronically it should be possible to actually sound the ideal structure of any type of instrument.

Since we know what we are looking for it should not be difficult to physically analyze the sequential structure of any instrument, including, the voice.

THE CARRIER STRUCTURE

Webster defines sound thus: 'mechanical radiant energy that is transmitted by longitudinal pressure waves in a material medium (as air) and is the objective cause of hearing.'

This does not go against anything thus far presented. The diametric sequences are indeed longitudinal. We must pause a moment on the term 'pressure waves' (this being synonymous with 'compression wave'. We shall define compression as 'an increase in density effected by a reduction in volume.' This is rather obvious in a gas.

We have already seen that a gas, such as air, can behave much like a liquid in that it moves as a body, such as in the oscillation phase of vibration. It can also behave as a solid in that it can produce powerful longitudinal structures that are diametric in nature. In the airborne carrier wave we see a gas behaving as a gas.

This compressibility that so characterizes gases, and is unique to them, is what makes a gas a gas. If the diametric structure proves to be a constant occurrence (and, indeed, all sounds are generated by a diametric structure of some sort or another) it would certainly present a different possibility of explaining the transmission of radiant energy longitudinally. Fortunately, if our goal is to produce a music machine, it does not matter how a resonating body creates diametric nodes. All that matters to us is at what frequencies do they form and their sequential order, and this will be largely determined empirically.

It is also quite obvious that the diametric structures and the airborne carrier wave are structurally and functionally completely different. In most likely every case it is the sounding of the diametric nodes that creates the airborne wave. Consider then what the airborne carrier actually does. It picks up the structure at the interface across the proper section of a resonating body then physically transports that information across a given space, depositing it on the eardrum. This carrier wave neither generates nor interprets the sound, it merely carries it, and yet, it is this structure we nearly always refer to when we use the term 'sound'.

The more we look, the more places we will begin to see these diametric structures. When an object falls into a liquid a circular corona is formed. The corona retreats and a true diametric structure forms. There is a beautiful Edgerton high-speed film of a drop of milk where the diametric structure formed is clearly twelve equal, pie shaped sections.

This powerful structure is what creates the concentric rings. These rings transmit the information of the diametric nodes by periodic changes in the resulting waves. The waves carry the information created by the impulse function and the resulting crown-like structure behaves much as does the airborne carrier.

It is most likely that the sound of the splash is created by this diametric system. The diametric structure in a splash appears very briefly. Most high-speed photographs miss it entirely. The sound of a large splash most closely resembles the sound of a cymbal. Both are circular, which is indeed the musical shape. Both get their complex, 'noisy' sound from the abundance of higher frequency inputs. Both are pitched.

Because of its complex sound we can expect the sequence of the diametric nodes of a splash to be composed of fairly high input frequencies (relative to the fundamental). However, in spite of this complex input the ring itself appears, to a casual observer, as virtually sinusoidal.

The reason for this is quite simple; water is viscous. It simply cannot respond to the diametric patterns in the same time frame in which these patterns are generated.

This causes a lag and thus various groupings of the diametric nodes sound simultaneously. This, as we shall see, considerably changes the nature of the structure. As a result of this change the ring is a 'smoothed out' version of the diametric input. It is not that anything is really lost. All of the information that is in the diametric structures is in the ring itself. A closer look at the shape of the wave in the ring would show a wave that is anything but a simple sinusoid.

The airborne carrier is, in many ways, quite similar. Both are generated by the same diametric structure. Both transmit energy by a periodic and reciprocal change in the density of discrete volumes of a given medium (in this case, air or water). While the ring appears to undulate actually the only real motion is rectilinear. It can only move up and down. The air is exactly the same. Whereas the ring in the water is a gravity wave the airborne carrier is a true compression wave. The molecules in the air move, for the most part, rectilinearly, in one direction when the air is compressed, the other when it is decompressed.

Whether or not true compression waves form in fluids or solids is an argument that fortunately we can avoid. When we state that an airplane is traveling at the 'speed of sound' it is the transmission rate of this airborne carrier in normal atmospheric conditions to which we refer. Virtually all attempts at mechanical analysis use this carrier as the subject of the analysis.

As with water, air is viscous. Air is of sufficient viscosity so that it takes about six full cycles at middle **C (264Hz)** to initiate the structure. This lagging effect is known as hysteresis.

Because of the fact that the frequencies of the diametric nodes must be sounding sequentially, we can expect to find little or no hysteresis effect in this phase of sound generation (the nodal structures). However, when these patterns are fed to the air at least six consecutive periods of the tone at middle **C** (a period being defined by the length of the fundamental) are made to sound simultaneously.

The immediate result of making a horizontal,' sequential structure into a vertical, harmonic one is the emergence of a differential structure. This structure is completely absent in the diametric resonance. Without experiment it is impossible to determine just how complex a pattern will be necessary to create a given sound. It may be contained completely within one period. It is not likely to go more than six.

This very complex differential structure is what we observe when we feed the result of a microphone (reacting to the airborne carrier) to an oscilloscope. The patterns that created this 'sound wave' are still present but they are virtually incomprehensible. We then further complicate things by considering this to be a linear phenomenon. Any 'curved lines' present in an oscilloscope reading are caused largely by the hysteresis process inherent in the air itself and, to a lesser degree, to the hysteresis present in the magnets of the microphone and the oscilloscope itself. The oscilloscope is designed to produce a linear representation of the information fed to it. If it is fed a series of discrete points the oscilloscope connects the points and we are presented with a linear display of a definitely non-linear phenomenon.

Nor will direct analysis of this analog waveform yield any clues as to which of the sounding tones are from the initial input and which are differentially produced. Whether or not the patterns themselves can be deduced from the airborne wave, which we might actually call an amalgamated wave, must await experimentation. It may be possible to create an algorithm that will allow us to extract this information from the wave we see in an oscilloscope.

It seems counterproductive to use an infinite series to analyze a structure that is both discrete and finite. This structure is easily assessable mathematically. The concept of the type of thinking that accepts the sound wave as an analog structure manifests itself mechanically is a process known as digital sampling.

The main problem structurally with sampling is that this sampling imposes a number system upon the structure that has nothing to do with the mathematics of the structure itself. If our goal is to produce music then the wave we see in the oscilloscope is not what we are after.

It is true that ultimately, if it is to be heard, an airborne carrier must be generated and sooner or later we will be dealing with a very complex differential wave structure. We must not forget that this very complexity is caused by rather severe hysteresis induced by the viscosity of the air itself. When sound propagates in water or steel it does so as diametric frequency patterns that are much less affected.

We get this 'wave form' by changing the variations in the pressure of the airborne carrier into corresponding variations in voltage. These variations in voltage are usually thought of as being analog in nature. It is this particular notion that has given rise to the technique of digital sampling. If the sampling rate is set high enough it is true that a recreation of the wave seen in the oscilloscope can be created that is indistinguishable from the wave in the oscilloscope, however that wave is not what we want.

If, on the other hand, we consider this complex wave to be the sound it would be logical to assume that if we feed the fluctuations in voltage back into a transducer (similar, in many respects, to the one that changed the pressures into voltages initially) we should excite a structure in the air similar to the one that existed originally. This is the thought behind all electronic sound recording and reproduction.

In spite of many years of exquisitely sophisticated research involving many truly gifted people a practical loudspeaker cannot be made that sounds 'real'; sometimes very good, but not real. This has nothing at all to do with the electronics involved.

The speaker is a musical instrument and its evolution, like most musical instruments, was directed largely by trial and error. One thing nearly all speakers have in common is an air space. It is in this air space that the diametric nodes form that actually interfaces with the airborne carrier. As with all instruments this air space must get the patterns themselves from somewhere. (The patterns that form in the sound box of a violin originate in the string, etc.).

A diaphragm excites the diametric pattern within the air space of the speaker. This uses magnets and thus the structure is further corrupted by hysteresis. An even bigger problem is caused by the fact that the information fed to speaker is the result of an airborne carrier, and thus the signal is already corrupted. The speaker corrupts it again. This is why public address systems do not create a result that sounds real.

So long as we have the patterns we can inject the energy directly into the sides, just as the oscillation itself does. We will not use anything that vibrates, such as a diaphragm. These discrete patterns can be fed to a resonant chamber by much simpler methods.

This opens the door for many fascinating possibilities. We could, for instance, feed the patterns into the sound box of a violin. The input would be, of course, perfect, and the violin body would respond in kind. The possibilities of this sort of thing are virtually endless.

Once we know the patterns, interfacing them with the airborne carrier should be the easiest part.

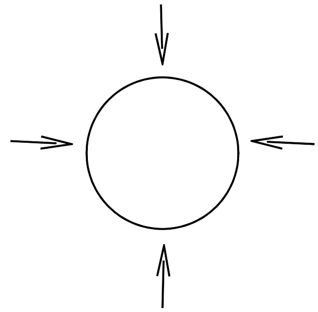
THE PATTERNS

This interface to the airborne carrier is the only mechanical part of the proposed device. It is possibly the most crucial. Even if we were to construct all of the necessary electronics correctly it would all be meaningless if we were to feed the results into an oscillating body.

The shape and construction material of these speakers will have to be determined empirically. There are too many factors involved to allow us to even speculate at this point. We can be specific about exactly what must be accomplished. A precise, discrete series of frequencies must be injected accurately and efficiently into the sides of a resonating chamber.

The shapes of the air-to-air interface areas of musical instruments are a result of the nature of the instrument. A simple circular area should suffice for any input structure.

While it is longitudinal waves we are after it is much easier to inject the energy via transverse waves. Most likely something like this:



If we use four separate energy inputs of equal power and arrange them as in the drawing, that is, two opposing pairs of inputs arranged on the perpendicular. Each opposing pair must be 180 degrees out of phase and the pairs themselves must be in phase with each other.

This would prevent any oscillation from occurring. The only structure that can form will be longitudinal and it will be created by the same patterns that are being injected. The proper diametric structure should then form in the air space of the bell (or whatever). This will then interface with the air we should have our 'sound'. Feeding the information from this device into a standard speaker (that is, one with a vibrating cone, or membrane) would be a waste of time. This is critical.

Let us concern ourselves now with the patterns themselves. It is these that we shall be creating electronically. It can be argued that all sound is product of an assemblage of diametric patterns.

The pitch of a musical tone is always clearly defined and thus we can be reasonably sure that all of the elements of any given pattern will line up or come together at the beginning of each period of the fundamental. It may be possible that at least some of the patterns will be appear to be extra-periodic, that is, the length of the pattern extends beyond the length of a single period. The pattern, in this case, would then be segmented and thus would be formed by a series of complete periodic sections that together would create a larger pattern.

This thought is supported by the fact that in many tones the amplitude of the various overtones does not remain constant while the tone itself is sounding, even if the input to the tone is consistent. The same thing occurs when the tone is allowed to decay. In the decay of any tone we can see from harmonic analysis that the amplitude ratios between the various overtones is not constant. All such decay is non-linear. We can therefore assume that these patterns will also be structural and strive for certain ideals of structure.

Let us consider another very fascinating aspect of pattern-generated sound, which is speech. We will consider singing to be the same, structurally, as speech and thus anything in this discussion that refers to speech also refers to singing.

There is a duality in speech that is perplexing, considering two simple facts, one: a vibrating body can create one, and only one pattern at any given time. Two: the ear can receive one, and only one pattern at any given time.

As to the first, the throat and voice box apparatus create speech, which in itself, is an instrument. It makes one interface to the air through the mouth (or, much more inefficiently, through the nose). If two people say the word "the", we hear two distinct things. One: The timbre of the particular voice (this sounds like John, that like Mary) no matter what particular word is being said. Two: Even among widely differing timbres we recognize, in each case, the word "the".

In order for the single voice structure to produce two distinct functions two independent pattern structures must exist. We can expect the voice pattern to be longer than the one that produces timbre. Periodicity is of not as great an importance in word patterns as the timbre provides the pitch. This is why the word structure system continues to work when the more focused and powerful sounds of the singing voice are used.

The human mouth, when fully open, presents a formidable area of interface to the airborne carrier. It is quite necessary to keep this area of interface as consistent as much as possible. Unfortunately, the creation of speech is caused by the physical shape of the resonating chamber in the vocal structure (the formant structures) and is inconsistent with keeping the mouth open for many sounds. One or another of these vocal dualities must be sacrificed. This is the singer’s greatest problem. In a device as described by this paper such a problem will be non-existent. How the voice actually creates these patterns is of no importance to the generation of sound. As with timbre itself we have only to find out what they are.

Once an effective technique for accurately measuring the diametric patterns in speech we will be able to extract the absolute pattern for the word "the" with any accent or intonation. We can apply this to any sound we wish, even a perfect Stradivarius. All of this is done and mixed in software. In the final step all of the information is fed to the transducer as one pattern. Even the amateur will have at his disposal the finest 'singing' voices one can imagine, and as many as he wishes. Such voices will never tire, never sing out of tune and never create a timbre that is less than perfect.

THE CIRCUIT AND ITS USES

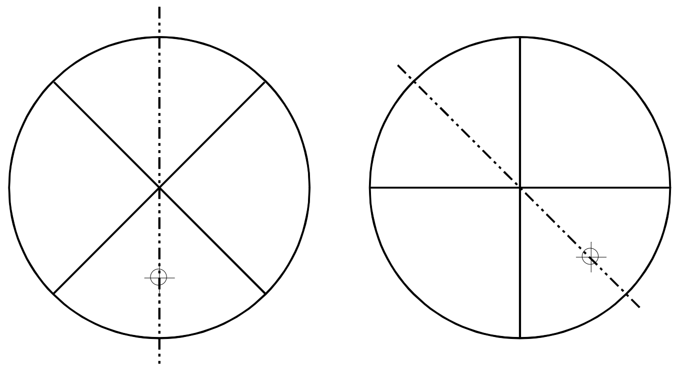
The first thing we will need is a topological look at the surface of the tympani as it goes through the production of a sound. Suppose we let the drumhead be tuned to low **F** (**88Hz**)

Since this is the third overtone the fundamental would be **29.3333Hz**. We would want to detect any patterns falling within the enharmonic matrix and it appears that the enharmonic rectangle defined by **135** will be sufficient If we multiply the fundamental by **135** we will need **3960** topological 'pictures' per second. This low number will allow us great latitude in testing the drumhead.

We must initially test the actual topography of the drumhead in order to establish exactly which patterns do appear. I propose that only the diametric patterns will actually appear but this must be established empirically. If the tympani proves this out we can be reasonably sure circular and composite nodes will not be part of the structure that creates a sound.

We must then establish the correlation between any given pattern and the sounding frequency. This should prove to be the same as we discovered in the Chladni patterns. If proven in the tympani we can assume this to be consistent with any instrument, and, in fact, any sound. We will then have but to seek the frequency patterns of the nodes. We will have a distinct advantage of knowing exactly where nodes and antinodes will appear

In order for the diametric nodes to appear the entrance of the energy must be off-center. This off-center point forms a line with the center that always bisects one of the pie shaped sections



In winds and brasses the plane of the reed will bisect the pie-shapes section as shown by the dashed line. We can expect the voice will divide the section across the plane of the lips, as with the brass.

Because of this the patterns would never be convergent. Once we established the bisecting line we will know exactly where each diametric pattern will fall on the body or end of the instrument itself.

The sound of a stroke on a tympani is very short so it will present no special problem to detect and record the entire sound as a series of patterns. Patterns that create timbre will probably cycle once each period and the length of the sound is certainly less than a second so we can expect to be studying the change, period to period, of less than **88** patterns.

If we multiply these tympani nodes by the enharmonic matrix we get the following,

792	**3960**	**27**	**135**
264	**1320**	**9**	**45**
-88-	**440**	**-3-**	**15**
29.3333	**46.6666**	**1**	**5**

This is achieved of course by multiplying the original matrix by **29.3333**.

Thus the patterns of the tympani sound a low F should be made up of the following series of frequencies.

792	**3960**
264	**1320**
-88-	**440**
29.3333	**46.6666**

The number of actual frequencies that make up the sound of a tympani stroke is quite small and easily manageable. Once we have this series we can do many things with it.

First of all we can feed it back into the tympani itself, bypassing the membrane. If this is correctly done the diametric structure should form, just as with the original sound. The airborne carrier will form in the same manner and the wave will affect the eardrum in precisely the same manner. The ear will be unable to detect a difference between the original sound and the reproduced sound because there will be no difference.

This modest test and experiment will virtually prove many of the premises of this proposal. Just on the tympani alone we can explore many things. The difference between a soft stroke and a hard one on the same pitch will mean a distinct change in the pattern structure. The distance from the center that is stuck by the mallet will alter the patterns. Raising or lowering the pitch will also change the patterns. By the time we go on to other instruments we will have a pretty good idea of what we are looking for.

THE ARRAY OF THE FREQUENCIES

The matrix is an array and as such it is very easy to modify all of the values of every number in the array (like multiply them all by **135**). All the values of the array are divisible by **3** and **5** and the result will always equal one. Once this multiplication has taken place the array will now be bounded by $\mathbf{3^{11}}$ on the x-axis and $\mathbf{5^{5}}$ on the y-axis. Simply put all of the matrix numbers can be created with a divide-by-n counter with a modulo of **3** and another with a modulo of **5**.

Multiplying the matrix by **135** allows for the pattern structure that will be needed to create the timbre of any given pitch. For instance, if we wish to sound the center of the matrix, which is **2025**, we would 'aim' at **273375**. That value would allow for the array that provides for the input of the patterns.

54675	**273375**
18225	**91125**
6075	**30375**
2025	**10125**

If we were to choose **10125** as a value to be analyzed we will find the it is created by four threes and three fives (**3*3*3*3*5*5*5 = 10125**). We will write this as **4.3**. Every number in the matrix can be designated in the same fashion. The list of these codes is all we need store or access. The program merely has to feed the output of the system clock into a divide by **3** counters four times and then through a divide by **5** counter three times. The result, since it formed directly on the clock frequency (which should be as fast as possible) will lie very high in the potential acoustic system. The result is looped through the divide by **2** counter until the desired frequency is achieved. All of this will take less than a nanosecond. The routines are so elementary that they will be written in Assembly Language. In such a case they will be virtually invisible, running at today's clock speeds.

Actually this description is useful for clarity but in practice we will use simple counters in a design that has been created. This will require no modifications such as octave shifts.

This method allows us to extend the enharmonic matrix as far as we wish. I can't help thinking that when we go higher in the matrix interesting things may be found that we are as of now unaware.

The function of the ear tells us that in the digital world, volume is a function of more signals per unit time. The ear responds to an increase in amplitude by sending more discrete signals per unit time on the various nerves. The sound generator will create any sound by a very complex set of instructions that do nothing but select numbers from the matrix codes at predetermined time intervals.

We state that a crystal oscillator pulses two billion times a second but the second is an arbitrary choice. It defines the rate of speed of the pulses, nothing more. The period, on the other hand, can be anything.

We shall assume the period be **553,584,375** pulses in length. We will call this the Enharmonic Period and abbreviate it EP. It can be used with any clock frequency and the length of the EP will be $\mathbf{553{,}584{,}375} / \mathbf{IP} = \boldsymbol{EP}$ where IP is the input frequency. Regardless of how many different divisions this EP is subjected to it will never suffer even the slightest jitter. Everything will always fit. We will do all of our dividing directly into the clock frequency itself. It can then be brought down into the acoustic world where it will be, for all intent and purpose, 'perfect'.

This device is not tied to the input in any way. No matter what the input frequency is the device will produce a perfect breakdown of the enharmonic matrix. A crystal oscillator whose output can be varied will give us all the control of pitch we might desire. These outputs will not only drive the generator itself but will drive the devices that will be necessary to find the patterns initially.

The fact that the computer developed, not for the undemanding needs of music, but rather for the insidious, myriad calculations required by such things as differential calculus is a definite plus for the design of this machine. Consider a concert pianist sailing through the **C♯** minor Etude of Chopin, just about the virtual limit to notes per second that can be played. Our pianist is playing about ten and the effect is, to say the least, breathtaking. In between each note a **2Ghz** clock will beat *two hundred million times*. This will allow us to write programs in a high level language that can be executed between each sounding note. This will allow us to modify the ideal partials so as to produce the comma modified structures we see in the 'Chladni patterns' of the various instruments should that become necessary.

Any given pattern will require a certain amount of data to define the necessary parameters the circuit needs to create the pattern itself. The pattern of a tone changes in response to a change in pitch. It also changes in response to a change in amplitude. Both of these can be easily calculated provided the processor has at least two reference points in each case.

Various types of decay will be programmable and achieved by alterations of the input patterns. One of the really difficult things for a synthesizer to achieve is what is known in music as an attack. Attack is the result of the impulse function necessary to initiate any sound. Digital sampling, which cannot produce a sound that is real, is even more ineffective when it attempts to synthesize an attack.

USES OF THE ARRAY GENERATOR

Strictly speaking music is a nested system. Like the great Samsara of Buddhism where the cycle of existences revolve around the human experience the music system revolves around the tone. The structure that creates the timbre of the tone and the structure that organizes it into musical pieces is the same in both cases.

Initially we have patterns consisting of discrete frequencies. Conceptually these patterns replace the now held notion that a tone is the result of a fundamental plus its overtones (usually sequential). This overtone structure we see in harmonic analyzers is a product of the airborne carrier and appears because of the nature of the air and has, in itself, nothing whatever to do with the creation of the structures themselves. It only serves to modify them.

It is quite possible that the middle ear developed just to remove this excess structure imposed upon it by the carrier. Thus the patterns that hit the Organ of Corti are, for the most part, exactly the same as those that appear in the original vibrating body. Studies have shown that the nerves that lie on the Organ of Corti in the cochlea transmit their information to the brain as discrete, non-linear pulses. As far as the cochlea is concerned the complex overtone structure that we see in the oscilloscope of the airborne carrier most likely is created by the nature of the air. What the brain actually does with all of this information is something else.

The patterns are all we need. What we actually perceive is a 'colored' tone. While the pitch may be the same the ear can easily recognize an oboe, a flute, a trumpet or Walter Cronkite.

The act of composing music consists of arranging these tones into various vertical and linear patterns. The choice of frequencies used in composing music is governed totally by the enharmonic matrix, just as are the patterns that create the timbre of the tones themselves. These in turn create differential structures that relate ultimately to one fundamental.

The studies of the Chladni patterns appear to indicate that the fundament of any tone is not actually sounded. The fundamental and the second overtone must be differential tones created by higher sounding partials. If the fundamental is sounded (as is done in all current synthesizers), the very sounding of this fundamental will corrupt the timbre of the differentially created fundamental.

Studies of the ear have shown that the sound of a tuning fork when it excites the Organ of Corti produces four overtones. Saunders suggests that this is a defect of the ear. After three and a half billion years of evolution it is hard to accept the fact that the ear would have such a defect. It is commonly assumed that the sound of a tuning fork is a ‘pure tone’ or sine wave. A pure tone is an abstraction. Just because we can visualize a sine wave in an oscilloscope does not mean that, when fed to a speaker, the carrier wave will contain a simple sine wave. Any sound must be the result of a structure.

At the other end is ‘white noise’. White noise is just as much an abstraction as the sine wave. It exists only mathematically. True white noise would contain an infinite number of partials and that is a physical impossibility.

THE ACOUSTIC ENVIRONMENT

One of the banes of a touring musician is the incredible vicissitude of the acoustics encountered as one moves from hall to hall. This is absolutely unpredictable. What the hall looks like, what it is made of, how big or small it is have virtually nothing to do with its acoustics. From the performer's point of view there is something almost gratuitous about the appearance of really good halls.

So much time and money has been expended on the field of listening environments that indeed the second definition of the term, acoustics is, 'the sum of the qualities that determine the value of an enclosure (as an auditorium) as to distinct hearing'. This is, unfortunately not necessarily the definition of a good hall. Distinct hearing is not enough, a hall with truly good acoustics provides much more.

We shall begin this discussion by reexamining a notion we touched upon in an earlier section. In any vibrating body, periodicity is established by the oscillation. In those instruments where the oscillation is air modulated (such as winds and brasses) the movement in the air is actually a compression wave. The proof of this is simple. The speed of sound is defined as being about **1050** feet per second. This is also the speed of the compression wave.

An open organ pipe four feet long sounds a note almost exactly equal to that of middle **C** in concert pitch (A **= 440**). Middle **C** is **264Hz** and we can see that $\frac{1050}{264}$ **= 3.97727273**. For this reason middle **C** is often referred to as **'four foot C'**.

If we lengthen the pipe the pitch goes down in direct arithmetic proportion. It would take a pipe some **38** feet long to sound the bottom note of a piano (the top note on the piano would require a pipe about a quarter of an inch long).

If we close the end of the pipe, everything is cut in half. Periodicity is determined by the time it takes the compression wave to travel to the end of the pipe and back. **Four foot C** takes a pipe two feet long. Changing the size of the diameter of the circular cross section does not change the periodicity, this being determined entirely by the length.

Instrument makers refer to the diameter of the circular cross section as the 'bore'. It is a very well chosen term. A change in the size of the bore changes the structure of the patterns and thus affects the timbre; it does not affect the pitch.

As we have seen, the cross section of the bore can be cubic or even rectangular as well as circular. While the change in shape affects the diametric patterns, pitch is still determined by the length. The nice thing about cylinders with a circular bore is that the length always remains the length. The direction of the length of a circular cylinder is always perpendicular to the plane of the circular cross section. Even when the diameter becomes much larger than the length, it is still the length that determines pitch. A closed cylinder, two feet long, will produce middle **C** regardless of the size of the bore.

The picture is not so clear when the bore is rectangular. So long as one dimension is much greater than either of the other two (as with 'square' organ pipes) there would be no problem. The long dimension would establish the pitch. As we enlarge the 'bore' in rectangular 'cylinders' it becomes more difficult to define the length. To be a good resonator the dimensions of a rectangular enclosure must be designed with musical proportions. The dimensions given by Vitruvius and others are just such. Before we explore shape as a factor let us take a closer look at a more direct function, size.

If we double the size of our two foot closed cylinder it will produce the **C** one octave below middle **C (132Hz).** Another doubling gives us **16 foot C** or a note equal to the low string of a cello **(66Hz).** This lies near the bottom of the practical range of musical instruments. Thirty two-foot **C**, while still discerned as a pitch **(33Hz)** would be used only for doubling sixteen foot **C**.

The only reason we would not use sixty-four foot **C** lies in the size of the pipe. No matter how you design it, a thirty-two foot pipe would be heavy and bulky, too much so to justify any strengthening the addition of a low **C** of **16.5Hz** would afford. The principle however remains unchanged. Let us imagine a closed pipe **128** feet long. This would give us **256 foot C** with a fundamental periodicity of **4.125Hz. 256** feet is not a long way for a compression wave to travel. For that matter, neither is **512** or 1**024** feet. We would never consider studying pipes this long but in actuality; a resonant, diametric structure would form in every case. It would seem that structures with fundamentals this low would be useless, but this is only true in organ pipes.

Let us consider a two-foot cylinder, closed at one end, with a square bore and a dimension of one foot in each direction. While this allows a complex timbre it still produces

four foot C as its pitch. The diametric structure is one-dimensional; it can form in only one direction. The same is true of oscillation.

We have a tendency to forget that vibration in an air chamber is a function of volume, just as with a solid. No matter how many dimensions the vibrating body might have, the body reacts as a discrete unit. For whatever reason, one of these linear dimensions will define the periodicity of the vibration. The reasons why and how this happens are complex and are not, in themselves, germane to this discussion. Empirical evidence shows this to be true. The body then reacts as a complex cylinder.

The more complex this shape the more complex is the diametric structure that creates the sound and the less musical it sounds. We usually define it as noise but we must be aware that the basic process that creates the sound is always the same. Noise is a subjective notion.

The one shape wherein this 'cylinderizing' might not occur is the sphere. In a sphere all dimensions are equal to the bore. We will take this up in a later section, as it has little use in the study of sound but offers fascinating ideas in some related fields. For now we shall concentrate on all of the other shapes.

THE LARGER STRUCTURES

Let us return to a two-foot closed pipe with a square bore equal to one-half the length. As with the circular bore we can double all of the dimensions, producing a fundamental one octave lower. Of course this lowers the entire diametric pattern structure as well.

If we continue this doubling until the length is equal to **128** feet, we will have a square bore cylinder with a bore of **64** feet. It will produce a fundamental frequency of **4.125Hz**. Such a large resonating body would actually constitute-a fairly large room. (As with all such calculations, I am assuming the speed of sound to be **1056** feet per second.)

It would take the compression wave about a quarter of a second to travel the length of the room and back again, but there is no reason to doubt that this would not set up the necessary undulations to create a diametric structure just as with a two foot device. Any discrete body, that is, a body composed of material of like impedance, will create one, and only one, fundamental.

It would appear that a resonance could form in any size of an object or a gas-filled volume. During the moon missions the LEM was crashed into the moon. The result was what NASA described as a resonance. Their description was that the moon ‘rang like a bell’. This causes us to question the notion that it is the shock or sound wave that creates a resonance. Such a traveling wave would take a relatively long time to bounce back and forth in order to set up the resonance. Since resonances can form in even larger structures we must reexamine the factors that create a resonance.

Since seems to be no discernable limit to the ‘size’ of a resonance we have to question the method of the formation of such a resonance. It may be that it is not the movement of a compression wave that sets up the resonance in either air filled areas or solids. A clue may come from a drop of liquid. The structure that forms in a splash as previously mentioned must come from the drop. This is a high-speed picture of a water drop hitting a surface. What it shows is definitely structure.

Quite likely this structure is the result of the surface tension that creates the drop. Surface tension in denser material is even greater. The surface tension of mercury is nineteen times that of water. Solids have surface tension as well. Perhaps the same thing occurs in a solid as occurs in the drop of water. This would mean that the structure that is excited by hitting or scraping it is already present. This would explain many things. It also suggests that there may be two different phenomena happening in an air space, regardless of the size of such a space. Along with the compression wave it may be that an intermolecular 'wave' is present that would be like the intermolecular wave in solids and liquids.

This could be determined by a simple experiment. We would use a piano with the loud pedal depressed and a microphone in one area. At a distance we would create a loud sound, perhaps with a trombone. We would then use sensors to detect the arrival of the sound wave to the open piano strings and the microphones. If there were a difference in the arrival time we would know there is two separate waves. This is highly speculative but possible.

Consider the small steel bars that create the sounds of a glockenspiel. The largest of these produces a high **G** of **792Hz**. The shape (a rectangular cylinder) is consistent and the surface is highly polished. This creates a simple, long lasting structure with amazing carrying power.

Suppose we double all of the dimensions of this bar and then double them all a second time. This would increase the mass **512** times. The weight of the bar would now be about **64** pounds and the fundamental would be nine octaves lower or about **1.5Hz**, certainly well below the range of hearing.

If we strike this larger bar (using, of course, a mallet of sufficient size) we elicit a sound that is quite audible. The proportions and construction material of both bars are the same and thus we can expect that the resonant structure that forms will be as well. It is, therefore, physically impossible for us to be hearing the same part of each structure. If it is the fundamental we are hearing in the smaller bar it cannot be the fundamental we hear in the larger bar. In fact, nothing much below the fourth octave (16th overtone) would be audible. To appreciate this difference, consider that if our fundamental is four foot, or middle **C**, the fourth octave is the top note of the piano (**4224Hz**).

The pitch of the clank from the **64**-pound bar would be at least seven or eight octaves above the actual fundamental. This means that the frequency the ear perceives as the principle tone, to use Raleigh's phrase, must lie between the **128th** and **256th** overtone. (The **128th** overtone of **four foot C** is **33,792Hz**).

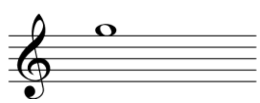
This explains why the small bar on the glockenspiel rings and the large, 64 pound bar clanks. The small bar, when struck, behaves like any impact-generated sound, that is, a very complex input structure appears immediately. The higher nodes of this initial structure, because of their higher energy demands, disappear rather quickly in a more or less top down fashion leaving a very stable standing structure composed of a symmetrical pattern structure drawn, most likely, from the ubiquitous six-four chord. Once the initial complex structure decays sufficiently (a small fraction of a second) the sound becomes very simple, consistent and quite lovely.

We would expect that the **64**-pound bar because of its greater mass (**512** times as much) would produce, structurally, the same thing, and, in fact it must. The large bar, when struck, just as with the small bar, produces at first a complex input structure, which quickly decays into a simple periodic sequence pattern. In the case of the large bar this entire pattern, as well as the differential structure it produces, is well below the threshold of hearing. What we are hearing when we hear the clank, is an internal differential 'build-up', a sub-fundamental so to speak.

The reason the clank is such a brief sound is that is part of that portion of the input structure that decays quickly. Another reason the sound is unique is that, while the ear perceives a pitch (from the **64** pound bar) to be somewhere in one of the octaves that surround middle **C**, it is a sub structure that forms high in the series and thus a series of tones below the perceived pitch will be sufficiently strong so as to be heard, in conjunction with the principle or audible tone.

This particular phenomenon that causes the clank only appears when the perceived pitch is sufficiently high so as to permit a complex overtone structure to form below it. As with all other music sounds, the farther away from the fundamental the perceived pitch is located the shorter and less musical will be the sound.

As we increase the size of the bar we would increase the size of the mallet as well, which would increase the amount of energy involved in the transfer. Sufficient energy would exist to boost even very high substructures into audible range. The lowest tone on a glockenspiel is a **G (792Hz** if **A = 440Hx**).



A cathedral bell sounding the same pitch weighs *six hundred pounds*. They cannot both be creating the sound by the same methods. The weight of bell tells us that it is not the body of the bell that creates the sound that we hear. When such a bell is struck there is a clank to be sure but then the characteristic sound of the bell takes over. This is created in the air with in the bell itself.

The Bessel shapes tell us that a pitch will only form when the sides of the cylinder are straight, they can conical but they must be straight. The part of the interior in that is straight in a six hundred pound bell is the correct length to form that **G.** As with all instruments the shape of a cathedral bell is constant**.** If we stand in the plane of the swinging bell we can observe that the bell sounds much louder when it swings toward us than when it swings away. Like most instruments the bell is a wind instrument. Its very name comes from the Anglo-Saxon word *bellan*, meaning to bellow or to make a noise.

The actual sound is somewhat more complex as the sound of the bell also contains what has often been called 'undertones'. This is cause by the fact that the tone we perceive is not the fundamental The bell is interesting to study, as it is one of the very few tone producing devices that involves very high energy.

I offer still another example to support this notion. For many years I lived by a very secluded lake. One very cold winter night I had opportunity to listen to the lake freeze. All natural sounds were dormant and civilized sounds were too far away to be heard and so I was able to hear some remarkable aspects of this freezing. As the lake froze the ice would swell and tremendous pressures would build up in the ice cover. From time to time this energy would be released in the form of huge cracks in the ice.

The initial sound of the crack was very sharp, much like a rifle shot. The sound source was localized, that is, one could determine, from the sound, exactly where on the ice the crack had occurred. As would be expected the more distant the sound was the less intense the apparent sound.

This very short crack was followed immediately by a definite, pitched musical sound. The behavior of this sound was considerably different from the crack that obviously generated it. The amplitude of this after-sound was quite the same regardless of the location of the input crack. It always sounded the same pitch and the same apparent loudness. This was especially exciting if the crack itself was at a distance.

All of the sounds of any particular freezing action were virtually identical. Every sound sounded exactly alike. The pitch was always the same; in fact, this pitch did not vary perceptibly from year to year. The amplitude of the after-sound was equally consistent.

The duration of the sound was also consistent. It lasted the better part of a second. This is a long time for a sound that is as complex as this one must be. The timbre, while always the same, is difficult to describe although on those very rare nights when the input energy was for some reason or other extremely high the sound was like the bellow of some prehistoric monster.

Consider the mechanism that is generating this sound. This particular lake is banana-shaped and is about a mile and a quarter long. It has a surface area of about one hundred ten acres. I can think of not one instance, in music, where water is used as a medium for generating diametric structures. It sometimes modifies the sound as in the ancient hydraulic organ but it is never used to generate a sound. On the other hand the potential inherent in water for the production of these structures is certainly no less than that of steel.

The lake itself is, in effect, a very large water drum. When frozen it has, as with any drum a stiff membrane across the surface. Most musical instruments require very little input energy. Higher energy devices such as the piano are already rare. Very high-energy devices such as bells are even more rare. As the mass increases the input energy must increase in like fashion. As the bell is enlarged the clapper is enlarged proportionately.

While the sound I was hearing must be many, many octaves away from the fundamental of that ice membrane (as it would determine pitch), in no way would the size of this lake limit the phenomenon. It is logical to assume that lakes much larger that this could create the same audible circumstances.

We create a sound in precisely the same manner as the lake when we drop a very cold ice cube into a glass of much warmer liquid. The transfer of heat between the water and the ice causes great imbalances in the temperature of the ice and the ice cube cracks, just as did Cranberry Lake. The sound that results is surprisingly loud and quite musical. It is one of those sounds that sounds pitched even though the pitch is virtually imperceptible (to most of us in any case). As with any most musical instruments the major interface with the airborne wave is between the standing structure that forms in the air within the glass and the outside air. Even the gentle banging together of two cubes or the cubes against the glass makes a familiar, musical sound that is easily heard above normal conversation.

Considering the volume of an ice cube to be a bit less than two cubic inches we can see that the ice cover of the lake, at an ice thickness of about six inches, would consist of something approaching two hundred million ice cubes. The energy levels that must build up to cause the cracks in this lake cover are awesome. Even a large cathedral bell is very small by comparison.

We would expect that this enormous energy input would produce a very powerful sound, and indeed it does, however, virtually all of the structural action lies well below the threshold of hearing. The fact that one hears such a consistent musical tone clearly indicates that even a lake the size of this (**110** acres) produces one, and only one structure (thus only one fundamental).

If we cover the glass containing the ice cube the sound is greatly restricted. The same holds true if we close the cover on a grand piano. The sounding board of a piano is open to the air on either side but the enclosed air chamber is only on the top. When the cover is closed the sound is greatly reduced. The piano too is a wind instrument.

A final example comes from an article that appeared in Discovery Magazine (April 2003, page 12). It described two fairly recent earthquakes in Mexico. The two were very similar in size, about **8.1** Richter. The first hit Mexico City with dreadful results. The second hit somewhat later and had the same Richter number and killed just two dozen people and created much less damage. It turns out that the land the Mexico City is fairly consistent as far as density goes and, according to a Mexican seismologist, Cinna Lomitz *it is pitched*. He states the pitch to have a period of **2.5** seconds.

In order for this to have occurred the earthquake must also have been pitched and its pitch was close to that of the land under Mexico City.

THE ENVIRONMENT AS STRUCTURE

A truly fine concert hall is consummate joy. A truly fine concert hall is also surprisingly rare. Architectural acoustics has become a defined discipline in its own right and yet very few halls can be classed as good, even among the newer halls.

The qualities that make up a really good hall are perhaps best understood from the point of view of the performer. Except for those instances of very good halls, wherein everything is special for audience and performer both, how much, or how clearly the listener hears is usually a function of where this listener might be sitting.

The major thrust in concert hall design in recent years has been centered on two concepts; a, the sound must be reflected in such a manner as to insure that every seat, as much as possible, receives the same basic mix and intensity of the airborne sound waves and b, all 'disturbing' echoes must be removed (this is usually referred to as excess sound).

This would appear to be logical as indeed, in bad halls conditions often exist wherein one seat will hear quite clearly while another, only a couple of seats away, will hear virtually nothing. Concert halls are, relatively speaking, small and thus echoes are very short. In bad halls these can be devastating to performer and audience alike.

An entire branch of architecture has arisen with its absorption factors and ray acoustics and, without question they have done their job well. Halls can now be designed to be as anechoic as the designer wishes. All permitted reflections are spread reasonably evenly among the audience and the materials used in the construction of the hall absorb much of the sound. The overall quality of this sound is something else again. The experience of performing in a hall where the ‘excess sound’ has been removed is very upsetting. Any performer relies upon the throwback of the sound to evaluate his or her performance. Without the resonance of a good hall it becomes especially difficult to hear other performers on the stage, even if they are very near to one another.

The first quality a singer or instrumentalist usually experiences when he or she encounters the infrequent excellent hall is that it takes so much less energy to perform. This is not an illusion. Neither is singing in the shower. Very little excess sound is removed in a shower stall. We sound good when singing in a shower because of the resonance that forms with the shower stall. Just as with the body of a woodwind the stall accepts any pitch.

As long the resonance forms most of the energy the singer ingests into the shower reinforces the sound. It takes very little energy to create a quite powerful sound.

Even when the sound is not of the same structure as the shower stall the echoes formed are so short as to create little disturbance. A room or hall constructed with the same materials as the shower stall is not so forgiving. In this case the echoes can be devastating. Unfortunately this is the phenomenon that is usually studied in conjunctions with concert halls.

A singer, singing in a fine concert hall, will experience exactly the same type of support as the shower stall singer. Singing in a bad or even mediocre hall is very hard work. Halls, which have been deadened by removing the ‘excess sound’, are just as hard. Performing in a hall that offers no throw back of the sound is a strange and unsettling experience.

In a really good hall the problem of where the good seats are disappears. Sound distribution is amazingly uniform within a resonance. Perhaps an example will help in understanding both the qualities of good and poor halls as well as the capriciousness of their appearance.

Several years ago I had the opportunity of performing the same opera in a great variety of performance areas. It has been my experience that if all else is proper the absolute best material for a concert area is stone, without qualification. Modern designers gasp at the thought of building a concert hall out of stone, and for good reason. The exquisite sensitivity to the nature and design of musical instruments that developed in our culture over the centuries unfortunately does not seem to exist in structures that form in anything larger than large organ pipes.

I will use as example two very large stone structures with extremely different characteristics. The first is the Cathedral of Saint John the Divine in New York City. In all fairness, St. John's was not designed to be a concert hall but cathedrals are among the world’s greatest natural theaters and thus the desire to make music in them is natural enough.

Since my group was the first real theatrical ensemble to ever invade the Cathedral, the Cathedral itself was in its pristine state. No attempts to reconcile those acoustics with performance realities had been made. The only logical place to perform was the crossing and thus all reflecting surfaces were a considerable distance away. This is the only significant difference between a large performing area and a small one. Apart from the variation in the return of signal all else is quite similar in both larger and smaller halls.

The first problem confronting a singer or instrumentalist is the amount of energy needed for the performance. Performing in St. John's, as with all such poor acoustical conditions, is very hard work. It requires considerable energy input to create and support a good sound.

Hearing presents an equally serious problem in poor acoustics. Performers will often state that they can't hear themselves. Actually, because of the resonance of the bone structure in the body, one can hear oneself, even in a totally anechoic chamber. A performer uses the throw-back of his sound to judge its quality. The throw-back he or she seeks can occur only under very special conditions.

This same throw-back of the sound is vital to ensemble performing; in fact perhaps it is this that is the worst aspect of the poor acoustical hall. In an area the size of St. John's this can be very frustrating. All of the magnificent playing area this Cathedral offers is neutralized by the acoustics. If a performer is more than six or eight feet away one simply does not hear him or her. Even if the performing group is shoulder-to-shoulder and not very large, music with any complexity at all must be conducted.

Large or small, these effects occur in any hall whatever if the acoustics are less than good. Everything about a bad hall is problematical. It is difficult to produce a tone, difficult to hear what is occurring and very hard to concentrate. I mention this in detail because there are conditions wherein none of these problems occur.

Some months after the St. John's performance I had the chance to perform the same opera with the same complement in another very large cathedral-like structure, the Chapel of the Resurrection in Valparaiso, Ind. It was nearly as large as St. John's and its reflecting surfaces are, as with St. John's, largely stone and glass.

Expecting the worst I had a vocal quartet sing, from the crossing, a very polyphonic section from the opera. What happened was just about the most beautiful sound that any of us had ever heard. It was truly awesome. Anywhere one went in the body of the church one experienced the same sound, and this, when the chapel was empty. There was no variation, anywhere. I found that I could place the singers as far apart as I wished, it did not matter, they could hear one another as well as when they were bunched together.

We had all experienced halls that produced this phenomenon but never on this scale, and in stone. The effect of such acoustics on a performing group is hard to imagine. We could spread out the production as much as we desired. A performer felt the same support from the church anywhere in the area and could hear all other performers clearly, no matter where they were. Because of this support everyone sounded glorious, to themselves and audience alike and performing was a true joy.

It is considerably easier to sing or play in tune in a good hall than in a bad. This is the biggest clue to just what it is that causes these discrepancies.

The only difference between these two churches was size and shape. Absorption factors and the like had nothing to do with this acoustical difference. We have forgotten that halls of any kind are, in themselves, musical instruments and obey exactly the same physical rules. The very same thing happens in a concert hall as happens in the air chamber of a trombone. We should expect a consistency of size and design that we simply do not see in modern theaters.

The two performances were consistent in all respects, including one that has become so axiomatic that we scarcely look at it and yet, it is the culprit in this entire good-bad hall syndrome. As with virtually any musical performance anywhere in the world we unthinkingly observed, *the pitch.*

PITCH STANDARDIZATION

There is no doubt that conventions are useful in most communication. Many things would be very difficult without them. Design of any kind would be nearly impossible without a universally accepted system of measurement (which is, of course, arbitrary). The danger of this is that often the mathematics of the measurement system and the numerical system of the function in question are not compatible. The very rigidity of the convention will usually obscure the nature of the problem.

The standardization of pitch as exists in the world today is the result of the development of instrumental music. Standard pitch of any kind has no use in vocal music. In an ideal situation the singers would use the pitch of the area in which they are performing. Unfortunately this does not happen in today's musical experience. The voice is simply not pitch restricted, as are most instruments.

The pitch of an instrument is, on the other hand, totally dependent upon its size. Instruments are therefore consistent; all **B♭** clarinets have identical bores and length. Manufacturing reasons prompted the adoption of **440Hz** as '**Concert A'**. There is no more reason for **440A** as for any other frequency and yet, we are taught, from the beginning, that ***A IS 440***. In music circles especially this pitch is held to be as immutable as the atomic weight of hydrogen.

When my group played in each area the performance was always preceded by a ritualistic 'tuning up'. Almost any performing group one might imagine will do the same thing. Someone will always have a tempered pitch pipe and before any music making occurs someone will sound the pitch. This is true even in vocal ensembles where a standard pitch is not necessary. If we called in a string quartet to test the acoustics of a new hall the first thing the quartet would do, without question, is tune to **440A**. We would be surprised if they did not and yet, this is precisely what it is that goes wrong in halls with poor acoustics.

The ancients seemed to know this and in fact there is a magnificent theater, built maybe twenty five hundred years ago that may prove this.

THE THEATER AT EPIDAURUS

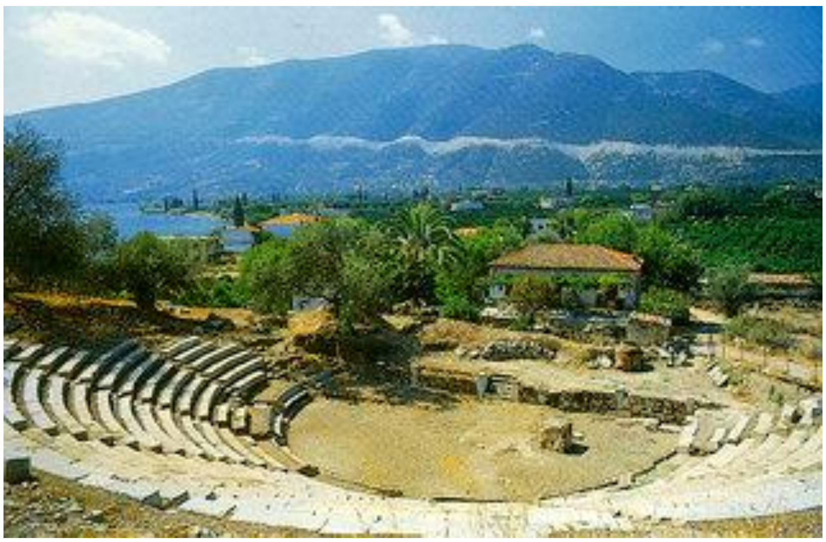
One of the truly frustrating things about the study of history is the difficulty in understanding the nature of the music of earlier cultures. Music is nearly always transmitted within a culture by rote. Ours is certainly one of the few cultures that have attempted to write about music in a language that is not musical in nature. Not only has little pertinent information survived but there was, in most cases, precious little written down in the first place.

Another difficulty arises. We can only view the culture in question through very modern eyes. The mistakes we make in contemporary understanding of music and acoustics we usually make when analyzing ancient systems.

On the other hand, the physics of music have not changed in the least. What is true today has always been true. To that extent we should be able to understand the music of any culture.

For centuries both the Greeks and the Romans built theaters with a very unique, consistent structure. The truly fascinating thing about these structures is that they are basically circular. A beautiful and amazingly in tact structure of this sort exists in Epidaurus. This structure might be sufficiently in tact so as to still perform, to a great degree, the function for which it was designed.

The structure consists of a circular stage area surrounded by a circular step-like structure. This step structure covers more than a semicircle and the face of each successive step (which is the area that defines the circle) is concentric with the circular stage area. The distances of the steps from the center of the structure appear to be in harmonic ratios. To my knowledge no precise measurement of distance of each individual riser from the center has been made.



This was a common design and many were constructed, first by the Greeks and subsequently by Romans.

Most contemporary acoustical physicists, when they consider architectural acoustics will speak of the 'problems' of the Greek and Roman theaters. The risers, they say, create debilitating echoes. This description of the acoustics resembles the description of the acoustics in a poor concert hall. Both the Romans and the Greeks built a lot of these structures over a period that lasted for centuries. They rank among the greatest architects in history and it does not seem likely that they would continuously construct structures this large and complex if they did not work, and work very well.

In preceding sections I have put forth the argument that the structure formed in a concert hall is identical to that which forms in virtually every instrument we use. Concert halls are, in and of themselves, musical instruments. The design of these Greco-Roman structures more resembles a musical instrument than anything else. We have observed earlier that nearly all musical instruments are cylinders with circular cross sections. These ancient structures are somewhat drum-like, or perhaps, bell-like.

The complete circle is not needed as the reflections from the risers would all be focused toward the center. A conjugate would be returned to the risers. In this way a standing structure would form, just as in a good concert hall or a musical instrument.

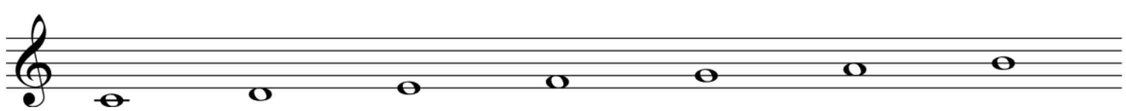
If it is true then that these circular structures are, in every way, musical instruments, and were designed to be such, then it must be true that the structure itself will respond only to a very precise set of frequencies, just as does any musical instrument.

Let us assume that we wished to test the existing acoustics of the theater at Epidaurus and to this end we invite a string quartet from a nearby university. The first thing this quartet would do, before it played so much as one note, would be to tune all of the instruments to the concert pitch, **440Hz**. We would be quite surprised if they did not. Virtually any group we might present would do exactly the same thing, this ritualistic sounding of the pitch. Even groups that do not need such tuning (such as vocal groups) will go to great pains to sing in the 'proper' pitch.

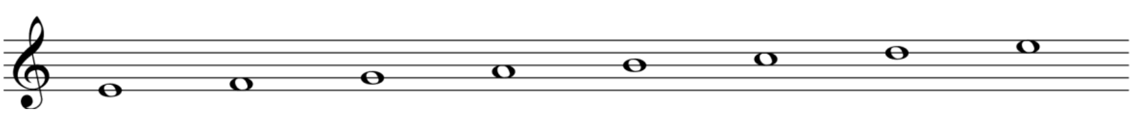
It is logical to assume that these ancient circular 'theaters' were in fact resonators. They are musical instruments in their own right and were designed to be such. As with any musical instrument there will only be a very few frequencies involved in the creation of the standing wave. The chances of **440Hz** or any of the scale tones generated from this pitch being one of the frequencies of the resonant structure of the theater at Epidaurus is extremely remote. So ingrained is the notion that **A** is **440** and all music must conform, that there seems to be no awareness of the problems this particular rigidity creates. We merrily go on using **440A** as a defining point and then talk about the problems of Greek and Roman theaters.

It seems that all discussion of the acoustics of Epidaurus speak in very general terms. The acoustics are tested by clapping or singing. Clapping is a complex impulse and may call forth the pitch of the theater but it is not particularly dependable. Singing would only be useful if the pitch chosen was the pitch of the theater. If such studies have been done there appears to be no data available about them. The actual pitch of any structure would be difficult to ascertain if one were not looking for it.

As with most music of the world, there was absolutely no musical reason, in the Greek or Roman world, to change tonality once such tonality had been established. Both the Greeks and the Romans used the modes, which are diatonic and relate to an unchanging fundamental. In its simplest form a mode is the diatonic scale, beginning and ending on a note of that scale. Consider: the major scale is; do, re, mi, fa, so, la, ti do. Phrygian mode would be; mi, fa, so, la, ti, do, re, mi.



Diatonic scale



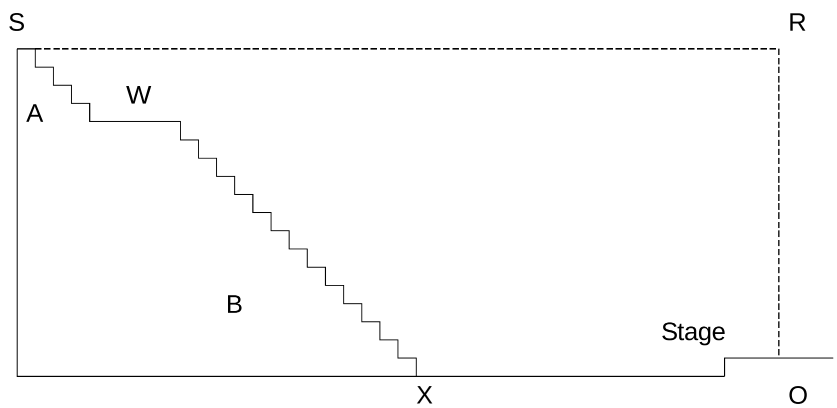
Phrygian mode

Actually modal structures are more complex but even this simple illustration shows that any given fundamental can support at least seven modes. These seven tones are all that the theater would be required to sound and all would be partials of the fundamental of the theater.

Being locked into one tonality would be then, no problem for an ancient theater. The structure at Epidaurus would have one, and only one, fundamental. Since the modes themselves are permutations of the same structure they, in turn, have a common fundamental. Poetry readings and speeches were very often sung in the ancient world. It is quite likely that the pitch they used was the pitch of Epidaurus itself. There is a theater in Athens whose acoustics have never been rivaled and where productions occurred that must be called operas.

There are two forces that mold any edifice, design and function. The consistency in design of both Greek and Roman structures would lead us to believe that these structures were very functional in nature, the design being almost totally dictated by what they do. In the case of these old structures, this function was to resonate. The ideal design for any musical instrument is circular, and the function of any musical instrument is to resonate.

The prime function of a resonator, such as Epidaurus, is to create a diametric structure, and this is exactly what the circles were designed to do. The mathematics of what happens with the structure must await some delicate and precise measurements that, for the most part, probably do not exist. Consider a simplified cross section of this structure at Epidaurus.



In the very center is the 'stage', whose center is pierced by the line RO. Section B is a series of what appears to be thirty-five risers. At W there is a very wide riser that creates a large horizontal area. At the far side of W is a second series of twelve risers.

This structure is not so easy to understand as it might at first appear. Is RS the bore or is OX? The length will be either RO or RO plus the thickness of the stage. We should be able to assess the fundamental of such a structure empirically (especially if we have the proposed device) and then tally up the math.

If the risers themselves represent simple overtones then the risers closest to the stage are the highest of these overtones. As the angle between the stage and each successive riser increases, less of the vertical face of each successive riser is accessible from the stage.

The first question that arises is why did they not complete the circular structure? Perhaps it wasn't necessary. Circular reflectors would reflect all sound to the center point. Any segment would do the same. When the reflected sound hits the center point a conjugate must be thrown back. A certain amount of sound would escape it is true, but much of the sound structure would be thrown back against the circular segments creating the standing structure in the volume of air within the theater.

In all probabilities the entire circular structure would form sympathetically, giving us an air-to-air interface not only on the top but on much of one side as well.

VITRUVIUS

The best source of information on Greek and roman Theaters comes from Vitruvius Pollo who was a major Roman Architect in the first century A.D. He is often erroneously said to be the first architect. He wrote extensively about many edifices that were already very old at the time of his writings.

These books were the bible for all architects until the Italian Renaissance. The reason tat they survived at all was because of the Scriptos of Charlemagne. An educated person in the ancient world was expected to be well versed in many different fields. In the introduction to book one he lists the things a good architect should know.

> He should be a good writer, a skilful draftsman, versed in geometry and optics, expert at figures, acquainted with history, informed on the principles of natural and moral philosophy, somewhat of a musician, not ignorant of the sciences both of law and physics, nor of the motions, laws, and relations to each other, of the heavenly bodies.

He detailed plans for various buildings are in the first four books. All of his proportions are composed of simple integral relationships, the simple ratios of music, even when the effect is for the eye.

A bit later in the article he continues:

> On this principal of arrangements, the voice, uttered from the stage as from a center, and spreading and striking against the cavities of the different vessels, as it comes in contact with them, will be increased in clearness of sound, and will wake an harmonious note in unison with itself.

Modern acousticians seem to reject these ideas, although it is hard to imagine why. Vitruvius' writings are excellent, even by today's standards so it is hard to imagine why Vitruvius would mention this if it were not true. It has also been said that the vases would only absorb sound and not reinforce it. This I find hard to understand. Such vases would boost the sound as well as a sounding board or resonant chamber of any sort.

In an article in the November 1963 edition of Scientific American Vern Knudsen wrote:

> Vitruvius, the first-century Roman architect and engineer, wrote that large vases tuned as resonators were often located in the seating area to reinforce certain sounds. Whether or not such vases were actually used is uncertain, but in any case they could only have absorbed sound, not reinforced it.

This is a very common assumption and it clearly shows the lack of understanding of the nature of a resonance. A good singer can break a crystal goblet by singing the proper note. This requires a very good grade of goblet and a singer who can maintain a steady pitch. The goblet will resonate to any pitch. At one pitch, the fundamental of the goblet, a different reaction occurs. The resonance that forms will increase so long as the sound is present. It is just like Helmhotz' swing. Every cycle of the sound from the singer give the resonance of the goblet a push. It is this that causes the resonance to become strongest. The fact that is overlooked in modern acoustics is that *the energy that breaks the glass comes from the glass*. This is discussed in part III of these papers.

This is why the vases in the Greek theaters would reinforce and amplify the sound in the theater. To say that the vases would absorb the sound is totally absurd.

Vitruvius goes to describe in detail the construction and placement of these vessels.

> On the foregoing principles, the brazen vases are to be made with mathematical proportions, depending on the size of the theatre. They are formed so, as when struck, to have sounds, whose intervals are a fourth, fifth, and so on consecutively to a fifteenth. Then, between the seats of the theatre, cavities having been prepared, they are disposed therein in musical order, but so as not to touch the wall in any part, but to have a clear space round them and over their top: they are fixed in an inverted position, and one the side towards the scene are supported by wedges not less than half a foot high: and openings are left towards the cavities on the lower beds of the steps, each two feet long, and half a foot wide.

Vitruvius uses such detail as this in all of his directions on construction of any sort. He leaves nothing to chance and musical rations dominate many of them

Notice that he states quite unambiguously that the proportions of the vases are dependant on the *size of the theater*. This can means only one thing; *these theaters were pitched.* When you performed in one of these theater you would have had to use the pitch of the theater. These theaters were, in every way, musical instruments.

He uses as his musical theory source Aristoxenus, a fourth century philosopher and student of Aristotle. Almost all of Aristoxenus' writings on music are lost but many probably survived al least until the time of Vitruvius, judging from the references by Vitruvius. In music Aristoxenus held that the notes of the scale are to be judged, not as the Pythagoreans held, by mathematical ratio, but by the ear. He also held the theory that the soul is related to the body as harmony to the parts of a musical instrument.

Many of the modern attempts to decipher the little that still remains of Aristoxenus' work use the tonometric system. While Aristoxenus terms can be a bit murky the use of the tonometric system only serves to make them even more obscure.

Since Aristoxenus did not hold to the Pythagorean belief in intervallic ratios we have no way of knowing what he meant when he described the 'movable tones' of the tetrachord theories of ancient Greek music. It is totally unclear what he meant by quartertones.
Here is representation of the old Greek system in modern notation.

The five tetrachords (marked by brackets) are named Hypaton, Meson, Synhemmenon, Diezeugmenon, Hyperbolean. These are marked by half notes. These are the fixed notes of the scale. The quarter notes mark the moveable tones. There are seven permanent tones, discounting the low A, which was named Proslambanomenos. All of the notes are named but as they have no meaning to this discussion I have omitted them.

These eighteen notes make up a class. It is in the concept of the classes that the theory becomes difficult to understand. Suffice to say for now that half notes remain fixed regardless of the class.

He is very specific as to where the vases should be placed and of the pitches. To the ancient Greeks the only consonances were the perfect fourth, the perfect fifth and the octave. They did not think in terms of superposed intervals so they listed the perfect eleventh, the perfect twelfth and the perfect fifteenth as well. These six intervals were the framework of the music of that time.

Their tuning system was somewhat different. Vitruvius said that in small theaters a horizontal range halfway up the theater should contain thirteen equally spaced niches each with a specific frequency.

In larger theaters he specified it be divided into four horizontal areas, the top three all with thirteen niches and vases. He described the tunings of the three ranges in the terms of Aristoxenus and the Greek methodology. The bottom layer should be the enharmonic system, the middle range the chromatic and the top the diatonic, the same range as the single range of the small theater.

It is difficult to know what he meant by enharmonic and chromatic as the terms are different those we use today. Here is an example of the tuning of the three ranges that gives a good idea of the complexity of the tunings of the thirty-eight vases. Vitruvius stated that the there should be no center drum on the middle range as there is no other note that forms a natural concord of sound.

The top stave would be the tuning in a small theater with only one row of vases.

These theaters not only were tuned but also were actually very sophisticated musical instruments. The use of vases apparently was dying out in theaters in the city of Rome although they were still used in the provinces at the time of Vitruvius' writings but even as he mentioned this he stressed the need for following the dimensions he put forth.

One distinct difference between Roman and Greek theaters was that the performers in Roman theaters played on the scaena or stage whereas the Greek performers played as well in the circular area called orchestra. This would have given Greek performers better access to the vases.

Greek performers often used masks shaped as megaphones. A theater with acoustics as good as these must not have had the need to direct the sound to sections of the audience. This would have been unnecessary as, just as with the acoustics of the Chapel of the Resurrection on Valparaiso, Ind., the sound would have been equal in all sections of the theater.

I think that the performers in the orchestra wore the masks and they directed the sound, not to the audience, but rather to specific vases. The theater would already be sounding the fundamental structure of the theater. By directing sound to the proper vases certain acoustic structures that were sub fundamentals of the basic fundamental could be reinforced.

This would much like playing a theater tuned to C but reinforcing the tonality of D or A for instance. It is very much like the idea I present for a properly tuned listening area. This is a very sophisticated idea but considering the intricacy of the layout of the vases it is quite likely that this is the way they 'played' these theaters.

Generations of performers played in these 'theaters'. Over the years they would have discovered every little nuance and would have played these great stone instruments with all of the finesse with which one might play any instrument today.

The Ten Books of Architecture are worth reading. They offer us a excellent look at the life and thinking of first century Romans. Here is a section in which Vitruvius shows how little the life and fortunes of the artist have changed in two thousand years.

> The Delphic Apollo, by the answer of his priestess, declared Socrates the wisest of men. Of him it is said he sagaciously observed that it had been well if men's breasts were open, and, as it were, with windows in them, so that every one might be acquainted with their sentiments. Would to God they had been so formed. We might then not only find out the virtues and vices of persons with facility, but being also enabled to obtain ocular knowledge of the science they profess, we might judge of their skill with certainty; whereby those who are really clever and learned would be held in proper esteem. But as nature has not formed us after this fashion, the talents of many men lie concealed within them, and this renders it so difficult to lay down an accurate theory of any art. *However an artist may promise to exert his talents, if he have not either plenty of money, or a good connection from his situation in life; or if he be not gifted with a good address or considerable eloquence, his study and application will go but little way to persuade persons that he is a competent artist.* (Italics mine)

THE CAVE

Before we relate all of this to contemporary structures let us briefly take one more step back in time. It is to prove a point.

Caves are very significant in the collective past of the species. Even today we often refer to all of the people of the Paleolithic epoch as cavemen. Considering the choices available to Stone Age man the cave is not a bad place to live. Temperature is both constant and comfortable and they are relatively easy to defend.

There were drawbacks. The safe, comfortable ones were usually very dark. Attempts at lighting could easily fill the cave with smoke. In spite of this many large caves are known to have been inhabited for unimaginable lengths of time. Only very adaptable, ritualistic tribes would have survived such a restrictive environment.

Equally as troublesome but never mentioned is another difficult problem, acoustics. A cave that was actually inhabited would support a fairly high population density - children, babies, animals and more. If these sounds were permitted to be random the result would be absolute chaos and often probably was.

Whatever else it is, a cave is an enclosed air chamber. We can expect that it will resonate and that the larger, room-like areas would have one, and only one ostensible fundamental frequency. Sooner or later any tribe living in any cave would discover it. This would be motivated by survival. Caves usually have highly reflective walls and any sound ingested in pitches that are not the resonant few would create ghastly and highly disruptive echoes. Consider the sound of an indoor swimming pool full of noisy swimmers.

Once they figured out that if they only sang a few related pitches everything in the entire cave would have changed. It would be possible to communicate to any point of the cave by singing no louder than a mother would sing to her baby. Many instruments most likely developed as signaling devices. It is not inconceivable that music itself came from the same source.

How they would interpret this resonance is anybody's guess. Large caves would provide a vast integral overtone potential, allowing them to sound many notes that feed the resonance. We are so used to thinking of music through our linear language of words that we forget that music is completely different, that its existence does not depend in the least on this linear language. One does not need words to sing. It is quite possible, even likely that primitive man often developed complex and exotic music systems. There is no way we can ever know but I strongly suspect that the inherent sophistication of many of these old systems would astonish all of us.

We can only guess at the possible sophistication they might have achieved in so many, many generations. A child growing up in a cave would know the pitch of the cave from before he could remember. We can expect a Paleolithic axiom that stated: you cannot use the pitch of one man's cave in another man's cave. Every cave, no matter its size, had its own pitch and only a significant change in the interior volume could alter it.

Every one of the Greek and Roman 'theaters' had its own pitch as well and just as with their Stone Age ancestors, generations of children no doubt grew to adulthood with an intimate knowledge of the pitch of Epidaurus.

It is hard not to believe that there is much continuity between the timeless cultures of the Paleolithic to the volatile culture of the Greeks. We know there must be Stone Age carry-overs that exist even today. The mysterious quality of the numbers **7, 11** and **13** might have roots far older than we might dare to believe.

The most frustrating thing for all of us who would know the past is the fact that music leaves few traces. By the same token man has always been able to sing. For various practical reasons we have grown away from this type of acoustical experience. Such a pullback has not proved beneficial in the least, in fact, quite the opposite. Today we usually think of music in terms of other branches of science. We forget that until quite recently music was considered to be a science in its own right. We have, to a great degree, trivialized it and this is unfortunate.

Because it does not stare out at us as does statuary is no reason to think that music did not have a profound effect on the development of the culture itself.

French researchers Iegor Reznikoff and Michel Davois have recently shown that cave art may well have been used in rituals accompanied by songs or chants. The two studied the acoustic resonances of three caves in the French Pyrenees by singing and whistling through five octaves as they walked slowly through each cave. At certain points the caves resonated in response to a particular note, and these points were carefully mapped.

When Rezinikoff and Dauvois compared their acoustic map with a map of the cave paintings, they found an astonishing relationship. The best resonance points were all well marked with images, while those with poor acoustics had very few pictures. Even if a resonance point offered little room for a full painting, it was marked in some way - by a set of red dots, for example.

Cave resonances are not particularly rare. It is not likely that Cro Magnon would mark the spots with what are often elaborate drawings just to mark such resonances. Unfortunately the researchers who found this remarkable set did not list or probably did not ascertain the frequency of the sounds they used at the various locations. It may well be that what made this cave interesting enough to those early people to mark those locations was that the pitches were closely related. This would have made the cave extraordinary indeed. If singers were placed at the various locations and they all sing the proper pitch the result would be a very impressive resonance.

Stone is a very good resonator. Such a resonance may have been heard at great distances.

THE MODERN CAVE

Any room, any hall, any church; in fact, any enclosed air filled area will behave in the same manner. They are all musical instruments, even if not by design. Not all areas resonate the same, but that is true of all musical instruments.

Not only did many caves of the Paleolithic and the circular structures of the Greeks and Romans each have its own individual pitch but any enclosed structure does so as well. For some reason or other we seem to expect that this pitch will always conform to concert standard of **440Hz** for **A** above middle **C**. Every cathedral in Europe must have, not only its own peculiarities but its own distinct pitch as well, and yet, there is, almost without doubt an even tempered organ tuned to **440A** in every one of them.

The only reason that the Chapel of the Resurrection was different from St, John the Divine was that by sheer chance did it conform, at least to a degree, to our pitch. In St. John's it was not even close. It is hard to find another explanation for why similar structures are so capricious acoustically.

If we accept the fact that any air filled structure is a musical instrument it becomes easier to understand that the ratio of the size of any instrument to the size of the air volume in which it is playing is highly critical. If the ratios are simple, good acoustics will occur. If the ratios are oblique, the results can be dreadful.

This is true so long as it is the hall itself that forms the standing structure within the hall. On the other hand a sympathetic resonance is not dependant on the size. The air chamber of any instrument will respond the pitch of any input provided the inputs are of a certain character. This is characteristic of any resonance. It should be possible to create a powerful sympathetic resonance in any performing area even if it doesn’t conform to the natural resonance of the area.

This would change the experience of a concert hall considerably. Once a standing structure has formed and a performer is playing or singing in the pitch of the hall all of the energy of the resonance would reinforce all of the pitches of the performer. This would make any area behave, to a significant degree, like the wonderful acoustics of the Chapel of the Resurrection in Indiana., only more so as we could add as much energy as we wished. As long as the input frequencies were the proper frequencies for the area the result would natural amplification.

The acoustics of a hall would be independent of size and construction material to a great degree. The best input for a given hall would be determined empirically. The result would be infinitely better than using a public address system.

It is possible that a strong resonance in any given area would organize all of the ambient sound within the area into the resonance itself. This has many interesting possibilities. It would allow us to maintain concert pitch (**440A**).

It should be a more practical way of 'conditioning' a room in a home.

SUMMARY

Once the tenets of the preceding are proven we will be able to precisely define the difference between vibration and resonance. The understanding of this difference is critical.

The airborne carrier is created by a resonance and not by vibration. Once the vibrating body is excited it often becomes necessary to insure that none of the sympathetic resonance are permitted to vibrate. We see this on the sounding board of a piano on which heavy braces are placed across the sounding board to stifle any such vibration. This will be the case for many non-musical uses of the Structural Resonance theories.

Musical uses will include true simulation of any music sound. It will permit us to create listening areas of any size or shape that will be ideal. As a teaching instrument it will be without equal. Both singers and string players learn harmony, as a rule, at an even-tempered keyboard. With proper training the very notion of improperly tuned intervals will simply cease to exist. Imagine growing up in such an environment.

There are many possible spin offs that would come from this theory. Many will not be ostensibly musical as we are dealing with vibration and resonance. It is has been said in fact that the quantum states of an atom are musical in nature as are wave theories. Some go so far as to say that when you study quantum physics you are studying music. There are many other things that are not really so speculative. The design and functioning of the cochlear implant for one.

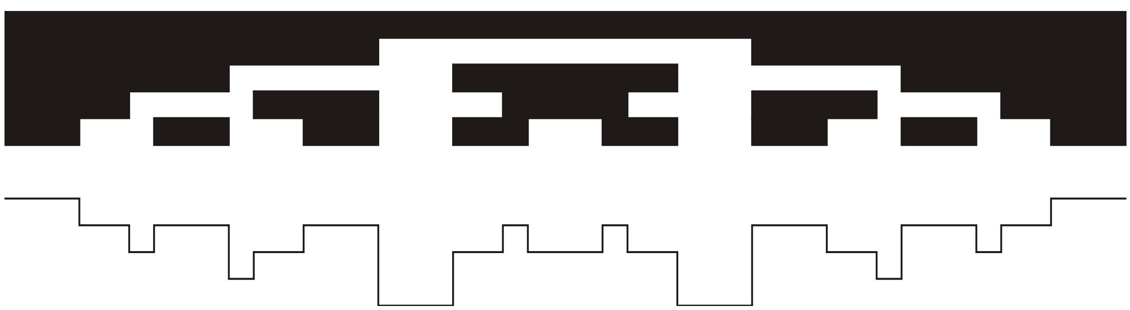
STRUCTURAL RESONANCE

THE MATHEMATICS OF MUSIC AND VIBRATION

PART THREE

CONTINUITY, MOTION and ENERGY AND OTHER THINGS

A true continuum is something
whose possibilities of determination
no multitude of individuals can exhaust
Charles Peirce



I can stand before the most august and erudite group of mathematicians and others of similar disciplines and say the following: *"Consider a line bounded by two points, A and B. Now consider the distance between those points A and B to be infinite".*

Instead of rising en masse to ask me if I have completely lost my mind most of the listeners will give that slight nod that signifies that they are considering just such a line. This is the true beauty of the human mind; that we can 'consider' a line infinitely long bounded by two points. We do it all the time. We have infinite sets, bounded and unbounded. We have unbounded sets which are much like our infinite but bounded line and we have empty sets which really take a bit of considering. And, of course we have irrational, transcendental and imaginary numbers.

In his delightful book, 'A Tour of the Calculus', David Berlinski states that if we deny the irrational numbers we avoid a very complex argument but then we have no calculus. This is really not correct. Whether irrational numbers are 'real' or not has nothing to do with the calculus. While calculus is used to describe various aspects of physical reality it, in itself, it has little actual relationship with physical reality. Bishop Berkeley and Leopold Kronecker both argued vehemently in regard to this and both were dismissed and misunderstood. Kronecker, when he said the God created the integers and man created everything else, was stating a simple fact. Only positive integers can really describe physical reality. All else is abstract and is invention, clever to be sure but an invention just the same.

This did not keep Kronecker from using these abstract fabrications quite brilliantly. He used such fabricated systems as calculus very well. Poincaré once stated that the problem with infinitesimals is that they are never quite infinite. This did not stop him from using the calculus to great advantage. The calculus needs no justification beyond the fact that it works, and it works amazingly well. Einstein put it best:

> *"As far as the laws of mathematics refer to reality, they are not certain; as far as they are certain, they do not refer to reality."*

We forget that even Euclidean geometry is abstract. Let's begin with the point. A point is meant to indicate location. As such it is said to have no dimensions. A dictionary definition will usually state that 'a point is *thought of* as being a location in space but having no dimensions'. The same dictionary will define a circle as 'a curved line, every *point* of which is equidistant from a center *point*'. A true circle, by definition, does not exist. It is an abstraction.

There is no reason why the diameter of a circle should form a rational relationship with the circumference any more than there should be rational value for the square root of two. If we follow the various mathematic devices used to develop a value for π we can come to the conclusion that a circle is a regular polygon with an infinite number of sides. That is as good a definition as any. We can add any number of sides to our polygon but we will never reach the circle – again the infinite series, the asymptotic function.

The world looks decidedly analog. So much so that it might appear strange that we would question it. In fact most people don't. Most scientific people will state, categorically, that the nature of the physical world is analog. The success of the mathematics of continuous functions is perhaps the greatest monument to the ingenuity of our species. From Newton's and Leibnitz' derivative of a real value function to Cauchy's development of the theory of limits to the field equations of Maxwell, this math has been the bedrock of scientific achievement. Without it our world of technical achievement and scientific understanding would not exist. Whether or not irrational numbers are real in such mathematics as this is without importance. It is a pointless argument.

The success of calculus is the prime reason we do not question the 'analog look' of reality.

When we watch movies or televisions shows we suspend belief in order to enjoy the imagination of the creators of such fare. We do much the same thing when we work with linear mathematics such as the calculus.

The first question is what is meant by real? You cannot define such a quantity as $\sqrt{2}$ numerically. We can create it symbolically. We can write $\sqrt{2}$ and treat it as if there were such a thing. The problem is that *there is no number which, when multiplied by itself will equal 2.* This is a fact. Any attempt to find the $\sqrt{2}$ always produces an unending decimal whose decimal part goes on and on and never produces a recognizable pattern. Certain fractions also produce unending decimals, such as $\mathbf{4/3 = 1.333333\ldots}$ or $\mathbf{15/28 = 0.53571428571428571428.}$ In the first instance a single digit is repeated endlessly and in the second the pattern **571428** is repeated endlessly.

These decimals behave differently from $\sqrt{2}$. If we multiply **1.333333** by **15** we get the digit **20.**

On the other hand with **0.535714285714285714 28** if we multiply this value with either of the members of the fraction we get the other member.

0.53571428571428571428 x 15 = 28
0.53571428571428571428 x 28 = 15

While this is painfully obvious it defines the difference between ration and irrational numbers. No matter with what you multiply an irrational number you always get another irrational number. When an irrational number is brought into any mathematic calculation absolute accuracy is lost. This is quite obvious in musical mathematics in which irrational numbers are an absurdity.

Kronecker also argued that negative numbers and zero are abstractions. Zero is strange fellow. It is also an invention, albeit a useful one. It is usually classed as a digit and a positive digit at that. It serves well as a marker in base ten arithmetic. It allows us to write such equations as **5 – 5 = 0.** Classifying it a digit requires a bit of imagination. This idea probably stems from its use to divide the positive and negative numbers. Negative numbers are usually described as separated from the positive by zero.

-5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5

The same thing can be expressed by shifting the **+1** to the **–5** location.

+1, +2, +3, +4, +5, +6, +7, +8, +9, +10, +11

Mathematically the two sets are equivalent however the second set avoids both the zero and negative numbers. This is similar to the expression of the enharmonic system as digits.

In set theory we have to deal with the empty set. This is another of mathematics oxymorons. On the other side of the coin is the infinite set. Set theory, as with so much of our mathematical thinking is often more philosophy than something that deals with physical reality.

It is convenient to use zero as the starting point of a Cartesian graph and zero can be used to mark the division between positive and negative numbers but even in this case counting cannot start with zero. In exponential systems, such as music, it is illogical even to start with **1**, as **1** with any exponent will still equal **1**. Using zero as an exponent is another abstraction, convenient but artificial. We get around the problem of starting an exponential series with **1** by the abstraction x^0. The exponential series of **3** begins with **3**, not **1**. Pythagoras' notion of the non-existence of the integers **1** and **2** suggests a sophistication of which we can only guess.

We not only define zero as an integer but as a positive integer and then go on to state that $0^0 = 1$, or at least many do. This clearly indicates how much of our mathematical definitions are fabricated or, at best, assumed.

How much more convenient binary arithmetic would be if, instead of using **0** and **1** we were to use **1** and **2**. There is no reason to start counting with zero in the binary system. We find ourselves numbering bits in a register not only from right to left, but designating the first bit as bit zero. It is quite confusing when you first start learning about computers to find that, in a given register, bit **3** is the fourth bit from the right. The same thing occurs when we enumerate components in a logic circuit. A series of flip-flops in a counter is nearly always numbered with the first flip-flop as zero. There is no reason to use zero in a binary system even as a marker. We do not make this mistake in music, which is also, in some ways, a binary system. There is no zero in the mathematics of music and there is no need for one.

Geometrically speaking a line is a continuum of infinitely small points, which means of course that there an infinite number of points on *any* given line. No one will deny that this is abstract thinking. Dedekind believed that a line could be cut at one of these infinite points, the famous Dedekind cut. He believed that it was not possible for the rational numbers to describe all of the points in a line. He stated that between the rational number were the irrational numbers.

There is no limit to the number of points the rational numbers can describe. Once again we have the asymptote. The number of these rational numbers can never be infinite. The continuum of irrational points on a line is a fiction just as is the circle, for exactly the same reason.

Dedekind offers an interesting proof of the existence of irrational numbers. He uses reductio ad absurdum to achieve this proof. Actually what he proves is that there is no number to fulfill the conditions he sets forth in the proof. How many times we begin a mathematical argument with the statement – suppose there is a number that ---- when our intent is to prove that there is no such number.

Absence of proof does not in any way constitute proof. This argument is used often. Religious groups will argue that it is impossible to prove that God does not exist. This is true but the fact that it is true does not establish the existence of God. You cannot prove fairies don't exist. Or ghosts or UFOs.

It doesn't matter in the least that irrational numbers are fiction. The mathematics that uses them is effective with or without such proof.

In fact the reason we do question the reality of this analog world is that continuity brings us face to face, among other things, with that mother of all of imponderables, infinity. We even have symbol for it. Creating a symbol for something like infinity gives an unreal existence to something that simply does not exist. Symbols can be a bit dangerous as it is so easy to begin to believe that the meaning behind the symbol is real even when it is not. We forget that something that is unbounded is not infinite.

While we may give lip service to that abstract nature of such symbols, especially in mathematics, it seems that we disregard the semantics of these terms. We continuously speak of them as if they are real. How we use these terms is a mirror of how we think about them and to a great extent on how we think about abstraction in general. If we refer to an abstraction as being real we come to think of it as being real.

Consider the common mathematical statement: *there are an infinite number of solutions to the equation* $x + y = 2$. Such statements are very common and yet they go decidedly against the very rigorous nature of mathematics itself. They imply the existence of infinity. Such a statement, by the way it is worded, is false. There is no limit to the number of solutions to that equation but that number can never *be* infinite. If a person were to obtain immortality he or she would simply never die, he or she would never become infinitely old. Life expectance would be unbounded, but in no way infinite.

Then there is this one: $x \to \infty$. Consider how we read that, 'as x approaches infinity'. The fact that we have a symbol for infinity gives it an artificial reality. The mathematician will defend the practice and state the use of convergence and divergence and limits and these are indeed very ingenious and quite useful devices but they still do not establish an actual existence of infinity. We simply cannot actually define something that isn't.

Carl Sagan, in one of his imaginative TV excursions, attempted a description of infinity. He started with the definition of a googolplex ($\mathbf{10^{100}}$). Even with such a large number, he stated, infinity is still a long way off. This clearly shows how that, even when we know better, we still conceptualize and speak about infinity as being real; as if it were something we really could approach, as if it had 'size'. Actually if you take a googolplex, raise it to the power of a googolplex and the then raise the result again to the power of a googolplex you would not only have an unimaginably large number but infinity would be just as 'far way' as it was when you started.

Continuity was the going thing in eighteenth and nineteenth century mathematical thinking. This was a time where the corpuscular nature of matter was in question and also the commonly held belief was that since light was a wave (continuity) there must be something to wave which gave us, of course, the concept of the luminous ether. It was also during this time that we substantially developed the concept of fields and the mathematics that describe them. We think of fields as being continuous because that is the way we define them mathematically.

Calculus was developed primarily to deal with continuous or constantly changing phenomena. Its usefulness cannot be challenged. How would we find the area under a curve without the calculus? As with any such powerful tool it affected our concept of the physical world. While references were made to lines of force, field theory reigned supreme. If you place some metal filings on a sheet of paper and place a magnet under the paper we can observe definite lines, which does suggest structure, and the mathematics of the continuous field does not necessarily define structure. In later times we see lines in solar prominences. They both have the look of something that is truly discrete. If the magnetic field is continuous why do we see lines? In a lecture in 1881 about the work of Michael Faraday, Hermann Helmholtz refers several times to the luminous ether. He did not qualify or question this in any way (neither did Poincaré. At this time in history the existence of the luminous ether was, for the most part, a given. This concept was to come crashing down in a short time after this lecture. About the time of this lecture it was the common belief that science had just about explained everything of importance as indeed fantastic things had been accomplished with this linear mathematics and the concepts it spawned. The Michelson - Morley experiment upset the cart and the luminous ether theory was gone and with it a lot of closely held convictions.

For a long time electricity was thought of as being a type of liquid. We still speak of the 'flow' of electricity or the 'force' of gravity. A few years before the Michelson - Morley experiment J. J. Thompson found the electron and the discrete nature of electricity was established.

Plank described the quanta and the foundations of field theory, which seemed so sturdy, began to show some cracks. Soon after that Einstein described the photon and it became quite obvious that the entire electro-magnetic spectrum was discrete, and as such, must be composed of structures. The subatomic particles that make up matter and the quanta of electromagnetic radiation are discrete even if they do appear to exhibit wave-like tendencies. The point of this is that even though this discrete nature is obvious, the math we use remains the math of continuous functions and it is the math that greatly molds our concepts.

An infinite series is an abstraction even though many mathematicians now tell us they can prove that the sum of an infinite series is **1**. This thinking, while perhaps useful in mathematical practice, is actually wrong. It is made absurd by definition. Since the series is unbounded you cannot physically sum the series anymore than you can approach infinity. This is the point at which we really step from mathematics into philosophy.

I once had friendship with a remarkable six-year-old girl. We became enmeshed in a discussion about the nature of infinity. What had started as a childish 'O yes you can' argument quickly became a real discussion of the nature of counting. After thinking about it for several days she dragged me off to the place where we always held serious discussions. "You can count to infinity," she said, quite seriously, "you would have to start counting now and count the rest of your life. Then keep counting after you die. You would have to count….." her mouth dropped open as the realization hit her. "You would have to count for - - *infinity years*." That is also how long it would take to sum an infinite series, proof or no proof.

That a series such as $(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots\ldots\ldots \frac{1}{n})$ is unbounded and, the fact that as it progresses, converges toward **1** cannot be challenged. We must remain aware that, like the true asymptote, the series never reaches its goal. If the series truly reaches **1** then there must be a penultimate fraction. If this fraction cannot be defined then the series never reaches **1**. The denominators of the successive fractions are integers that are increasing by powers of **2** and there is no limit to the potential size of these integers.

This argument is certainly not new. Supposedly irrational numbers were discovered by one of the Pythagoreans named Hippasus of Mentapontum. He was said to have discovered irrational numbers on a boating trip when he tried to express $\sqrt{2}$ as a fraction. Pythagoras was said to have believed in the absoluteness of number and so could not accept this and had Hippasus thrown overboard and drowned. If he had had the power, Kronecker would probably have done the same to George Cantor.

It has been said that Pythagoras killed Hippasus because he could not prove the non-existence of irrational number by logic. This is a flawed argument for the reason expressed earlier and I cannot imagine it having been made by Pythagoras.

We have to take this story with a grain of salt. Virtually everything we know about Pythagoras and his followers comes to us two or three centuries later and by such writers as Plato and later the Neo Pythagoreans. From the original Pythagoreans we have virtually nothing.

Pythagoras was one of the few outsiders who were exposed to the secret knowledge of the Egyptian priests. About that knowledge we know absolutely nothing. Our notions about the construction of the Egyptian pyramids is another example of how, when we have decided in our minds that something is true, we totally ignore the logic that shows that it is wrong. It is estimated that the construction of the great pyramid required two and a half million stones. It is also believed that it was constructed in something Like twenty-three years.

23 years is **8399** days, counting leap years. That is **298** stones a day, which works out to around twelve and a half stones an hour. That would mean a stone roughly every five minutes.

That is assuming they worked twenty-four hours a day seven days a week. Let's assume perhaps they worked ten-hour days. Then they would have needed to lay a stone about every three minutes, again working seven-day weeks for twenty-three years. A stone every three minutes is the figure that most Egyptologists use.

Think of the logistics of that. The stones would have to have been quarried, moved to the site, cut to shape and dragged up the pyramid. Unless they had a way to levitate the stones and direct them into place they simply could not have built that pyramid in twenty-three years.

They would have needed a technology that would have been close to magic. Without such magical technology they would have struggled to build that pyramid in a century.

If they did not build the pyramid in twenty-three years it certainly was not a tomb. Even the most altruistic Pharaoh would not build a monument like that for his great grandson.

Quite possibly Pythagoras' interest in the nature of the right triangle was that the three basic integers that are the primary solution of the theorem $\mathbf{a^2 + b^2 = c^2}$, those being **3**, **4**, and **5**, are three integers that are the foundation for the entire enharmonic system that defines all music. The development of the complete enharmonic system requires nothing more than basic arithmetic. It would have been well within Pythagoras' mathematical skills. Everything in music can be created by nothing more than multiples and powers of these three integers. Pythagoras felt that the system of number started with **3**, that **1** and **2** did not exist.

Let's take a closer look at the effect of irrational numbers in music.

THE TEMPERED SCALE

Music affords us the best example of this linearizing of non-linear phenomena. The twelve-tone system, which really only arose to satisfy the tuning problems of keyboard instruments, has become the primary tenet of contemporary music theory with disastrous results. One problem that has appeared on the music scene is that of hearing loss associated with the high volume many listeners prefer today. This is not so much the amplitude of the sound but by the deadly beat structures created by tempered tuning and the linear sound of modern synthesizers and reproduction systems.

Paul Hindemith once said that the ability to withstand tempered tuning is akin to the ability to adjust to pain. The same can be said about the synthesizer.

The dodecaphonic system has given birth to the tonometric system. By linearizing a non-linear system, in this case the chromatic scale, we have the notion that musical intervals can be measured and the elements on this linear yardstick are the cents of the tonometric system. The cent is the hundredth root of the twelfth root of **2**, $\sqrt[100]{\sqrt[12]{2}}$. A tonometric cent then is:

$$\sqrt[1200]{2} = 1.000577789506554859296792575793\mathbf{2}$$

The common defense of the tonometric system is that it offers a more precise way to *measure* intervals. This makes absolutely no sense and yet research papers in acoustics are rife with such references. $4/3$ and $264/198$ are both ratios that produce a decimal value of **1.3333333….** In this respect they are similar but can we state that the 'distance' between the numeric pairs that form these similar intervals are equal? If we let the digits of these fractions be cps we can see that for every cycle of **4** there are **66** cycles of **264**. In the tonometric system both intervals would be **498.045** cents, just as both just tuned intervals are decimally **1.333333**. The cents of the larger interval would be **66** times 'smaller' than the smaller interval. The tonometric system gives us nothing that we don't already have and obscures what we do have

Alexander J. Ellis, in 1885 translated Helmholtz' "On The Sensations of Tone". In it he converted everything into the tonometric system. Fortunately he did it all with footnotes so this edition still contains the Helmholtz original. The text of the footnotes nearly rivals the text of the original document. In a rather large appendix Mr. Ellis explains the tonometric system as well as why he used it. He stated that any music system must be related to the tempered scale to be analyzed. There is no physical justification for this statement. This was more than a century ago and the same thinking governs most of musical theory to this day as will soon be shown.

It is hoped that the Structural Resonance papers have established the fact that the tempered scale is acoustically, mathematically and musically useless. We cannot state that the tempered scale is wrong, as right and wrong have no meaning here. The tempered scale does exactly what it was designed to do, that is, to linearize the non-linear natural chromatic scale. As such it has no use in musical theory. It is a rather clumsy tool that has outlived its usefulness.

Mr. Ellis offers several methods of conversion to the tonometric system. In some of these he used logarithmic means. As with irrational roots, logarithms are of no use in musical or for the most part, acoustical theory. This preponderance of tonometric examples is unfortunate as it clouds an otherwise fine translation and some interesting observations made by Mr. Ellis. So often we find a paper on some aspect of musical theory that starts with an interesting premise only to see it lapse into the tonometric system.

A good example of this is in a study of ancient Peruvian pan flutes. The Quechua people created amazingly fine musical instruments. They made trumpets out of clay. This clay shrinks about ten percent when baked in a kiln. They made their instruments oversized by the exact amount necessary to allow for the shrinkage. The end results were remarkably in tune with one another. They used this technique to fashion pan flutes as well. Many of the pipes of individual pan flutes were tuned to very small intervals. The papers written by the musical anthropologists list the intervals in the tonometric system, which totally destroys any notion of what the function of these tunings might have had. The best explanation would be that they were used for comma displacement and, were that to be true, the ancient Peruvian music would have had a sophistication of which we are quite unaware. We can be quite sure that the ancient Peruvians did not use the tempered scale. All we need to know are the actual pitches of the pipes and these are never listed. We have to wonder how many other areas of study are clouded by this knee-jerk adherence to linear systems.

Here is a section from Mr. Ellis' appendix describing how to convert musical ratios into the tonometric system. After explaining how to raise a tone an octave by adding **1200** cents he offers this:

> 3. If the reduced interval ratio be such that 3 times the larger number is greater than 4 times the smaller, but twice the larger number is less than 3 times the smaller number, then multiply the larger number by 3, and the smaller number by 4, for a new interval ratio, and add 498 cents to the result.

He goes on to provide an example:

> Ex. For 32:47, then 3 x 47 = 141 is greater than 4 x 32 = 128, but 2 x 47 = 94 is less than 3 x 32 = 96. Hence we use the interval ratio 128:141 and add 498 cents to the result. If however as in 32:49, twice the larger number or 2 x 49 = 98, is not less than 3 times the smaller or 3 x 32 = 96, we use this interval ratio 96:98 or its equivalent 48:49 and add 702 cents to the result. In the first case the given interval having the ratio 32:47 lies between a Fourth and a Fifth, in case it is greater than a Fifth, but in both cases the reduced interval ratio 128:141 or 48:49 is less than a Fourth. The object of this reduction, which is seldom necessary, is to have to deal with ratios less than a Fourth. So much for the tonometric system.

A more practical method of converting musical intervals into the tonometric system is with the formula:

$$\log a \ * \ \frac{1200}{\log 2} = \textbf{interval in cents}$$

Where ***a*** is the decimal value of the interval to be converted. **1200** is used to define the octave. If we use the perfect fifth,

$$\mathbf{1.5,\ \log 1.5 = 0.17609125905568} \text{ and } \mathbf{\log 2 = 0.30102999566398}, \text{ thus}$$

$$\mathbf{0.17609125905568} \ \text{x} \ \frac{\mathbf{1200}}{\mathbf{0.30102999566398}} = \mathbf{701.955000865 \ cents}$$

Contemporary theorists will often define a deviation from a true interval in terms of the tonometric system. To state that an interval misses the perfect fifth by 4 cents says absolutely nothing. It would make just as much sense to state it misses the desired pitch by a ‘little bit’. This error, no matter how large or small will always be the ratio of two discrete pitches. It is physically impossible for it to be anything else. A perfect fifth is precisely $\mathbf{3/2}$ or **1.5.** It is not **701.955 cents.** Consider how the perfect fifth looks when written mathematically in the tonometric system:

$$\left(\sqrt[1200]{2}\right)^{701.955}$$

And even then it is not quite perfect. This works out to a value of **1.49999999925019893931034733767 58.** Of what use is a system in which the only interval that can be precisely defined is the octave?

Since so much contemporary material uses the tonometric system here is table of the intervals of the enharmonic system converted into their tonometric equivalents. The first column lists the true musical intervals as defined the earlier sections of these papers. The second column list the power to which $\sqrt[1200]{2}$ must be raised to approximate the true interval.

1.04166	**70.67243**		**1.48148**	**680.448715**
1.0546875	**92.178715**		**l.5**	**701.955**
1.066666	**111.731285**		**1.5625**	**772.62743**
1.111111	**182.403715**		**1.5802469**	**792.18**
1.125	**203.91**		**1.6**	**813.686285**
1.13777	**223.46257**		**1.6666**	**884.358715**
1.171875	**274.58243**		**1.6875**	**905.865**
1.185185	**294.135**		**1.706666**	**925.41757**
1.2	**315.641285**		**1.7578125**	**976.53743**
1.25	**386.313715**		**1.777777**	**996.09**
1.265625	**407.82**		**1.8**	**1017.596285**
1.28	**427.37257**		**1.875**	**1088.268715**
1.333333	**498.045**		**1.8962962**	**1107.821285**
1.35	**519.551285**		**1.92**	**1129.32757**
1.388888	**568.71743**			
1.40625	**590.223715**		**1.0125**	**21.506285**
1.422222	**590.223715**		**1.024**	**41.058855**
1.44	**631.28257**		**1.011358**	**19.55257**

The question that arises is; what can these cents be used for? I have seen endless texts with intervals listed as cents but I have never once seen them used for anything beyond the listing itself. Even if a use exists it would be valid only for the tempered scale.

Any variation in the intervallic ratio of the perfect fifth will always be rational. An interval is the result of two sounding frequencies both of which are, by definition, discrete. An irrational interval is a physical impossibility.

The bulk of contemporary thinking as to music theory comes from our musical calligraphy. If we play a fifth on the piano on the lower end of the keyboard and then play the same fifth on the higher end of the keyboard the two intervals appear to be the same size. This is also true in the calligraphy. A fifth in the bass clef is exactly the same ‘size’ as one in the treble clef. On the other hand, if a violinist pays a series of fifths his fingers will come closer together as the interval becomes higher in pitch. Both the keyboard and the calligraphy make a musical scale look isometric. One of the difficulties inherent in playing a violin is that as you play an ascending scale the notes, as fingered on the string, become successively closer together. This is not a linear change; the changes are discrete and they are not logarithmic. Therefore, far from being a precise “measurement”, the tonometric cent decreases in ’size’ as the pitch ascends just as does a rational interval.

When we speak of intervals as integral fractions we are speaking of ratios of course. While we speak of a perfect fourth as being **1.3333333…** it is really some octave of $\frac{4}{3}$, that is, $\frac{8}{6}$ or $\frac{64}{48}$ or $\frac{5784}{4338}$. The advantage of using the decimal is that it is always **1.333333…** regardless of the integers of the ratio.

Musical theory can be difficult enough to study without the tonometric system. Many elaborate theories have been proposed based on this system but the mere acceptance of tempered tuning obfuscates any musical aspects and makes the whole thing useless. A great example of this is the Schillinger System. Without complete acceptance of tempered tuning this system has absolutely no meaning. It reached its heyday in the mid twentieth century and has almost disappeared in recent years. The concepts upon which it was constructed are still very much with us today.

The Schillinger System reads more like a calculus primer than a method of musical composition. His entire system comes crashing down on two pages, the preface to Book V, the Special Theory of Harmony. This comes after 352 pages of rhythm and scales. He gives us Monomial Axial Combinations and Binomial Axial Combinations and Polynomial Axial Combinations and myriads of other similar things. We see many graphs and even the Golden Mean. He gives logarithms and geometric inversions. All of these are linear tools. He also states that there are 462 seven-unit scales. He then offers this:

> Chord-structures, contrary to common notion, do not derive from harmonics. If the evolution of chord -structures in musical harmony had paralleled the evolution of harmonics, we would never have acquired the developed forms of harmony we now possess.
>
> To begin with, a group of harmonics when simultaneously produced at equal amplitudes sounds like a saturated unison, not like a chord. In other words, a perfect harmony of frequencies and intensities does not result in musical harmony but rather in a unison. This means that through the use of harmonics, we would never have arrived at musical harmony. But actually, we do get harmony and for exactly the opposite reason. The relations of the sounds we use in equal temperament are not simple ratios (harmonic ratios).

The relations of the sounds used in equal temperament not only are not simple ratios but they are not ratios. Everything in music comes from harmonics. Mr. Schillinger makes the common mistake of thinking that music is something invented by man. Music is a natural event that governs all sound and, quite likely, all vibration. It is innate in man and shows itself in every culture with exactly the same structure.

He confuses the overtone structure of a tone with a harmonic structure comprised of separate complete tones. He totally overlooks the differential tone, which governs and structures all music. He forgets that a sounding tone and an overtone partial are two quite different things. It is believed that the partials are so called pure tones. This is not quite correct. Even if it were, it is impossible to create a pure tone acoustically.

The Fourier transform breaks down a complex wave into specific sine waves. This does not mean that the overtone partials are in fact sine waves. The overtones are part of a structure and do not exist as independent entities. If the partials were sine waves we should be able to take the results of such a transform (harmonic analysis) and by sounding the proper sine waves en masse at the proper times recreate a natural sound. That does appear to work, or at least not very well.

When we combine three sounding tones to form a triad we are sounding three distinct overtone series. If three flutes play a triad we hear a distinct sound.

They will play this triad amazingly close to the series,

6
5
4

Contrary to Mr. Schillinger's assertion they will never, ever play these approximations

1.4142135623730950488016887242097
1.3348398541700343648308318811845
1.2599210498948731647672106072782

If they played this tempered triad the three flutes would have the same overtone structures individually that they had when playing the true triad. If they attempted to play this tempered triad these overtones would interact with each other and form what can only be described as an acoustical mess. This of course happens every time with instruments tuned to the tempered scale. Can there be a more dreadful sound than Romantic pipe organ tuned with tempered tuning? Unless, perhaps an electronic organ.

Mr. Schillinger forgets that even the first theories of the tempered scale came rather late in the development of common practice harmony. Even during the centuries of plain song the practitioners of the art were well aware of the fact that some tones of the modal scales had more 'weight' than others. By the time of the baroque the chromatic scale was fully developed. The first true theoretical proposition for a tempered scale is attributed to Werkmeister in the 1690s' that was the time when the baroque was at its peak. Even Werkmeister's scale did not have isometric intervals.

Contrary to Mr. Schilinger's views the tempered scale had nothing whatsoever to do with the development of western harmony. There is no way such a harmonic system could have grown out of the tempered scale.

Here is the rest of the excerpt for the preface to Book V.

When acousticians and music theorists advocate "just intonation", that is, the intonation of harmonic ratios, they are not aware of the actual situation. On the other hand, the ratios they give for certain familiar chords, like the major triad (4:5:6), the minor triad (5:6:15), the dominant seventh-chord (4:5:6:7), do not correspond to the actual intonations of equal temperament. Some of these ratios, like $^7/_4$, deviate so much from the nearest intonation, like the minor seventh which we have adopted through habit, that it sounds to us out of tune.

Habits in music, as well as in all manifestations of life, are more important than natural phenomena. If the-problem of chord-structures in harmony were confined to the ratios nearest to equal temperament, we could have offered (16:19:24) for the minor triad, for example, as that ratio in fact approaches the tempered minor triad much more closely than (5:6:15). But, if accepted, this would discredit the approach commonly used in all textbooks on harmony, for the following reason: if such high harmonics as the 19th are necessary for the construction of a minor triad, what would chords of superior complexity, which are in use today, look like when expressed through ratios? When a violinist plays b as a leading tone to c and raises the pitch of b above the tempered b, his claims for higher acoustical perfection are nonsense as the nearest harmonic in that region is the 135^{th}.

Facing facts, we have to admit that all the *acoustical* explanations of chord-structures -- to the effect that they are developed from the simple ratios -- are pseudo-scientific attempts to rehabilitate musical harmony and to give the latter a greater prestige. Though the original reasoning in this field resulted from the honest spirit of investigation of Jean Philippe Rameau *(Generation Harmonique,* Paris, 1737), his successors overlooked the development of acoustical science. Their inspiration was Rameau-plus their own mental laziness and cowardice.

The whole misunderstanding in the field of musical harmony is due to

(1) underrating habit
(2) confusion of the term "harmonic" in its mathematical connotation -- i.e., pertaining to simple ratios -- with "harmony" in its musical connotation i.e., simultaneous pitch-assemblages varied in time sequence.

Thus, musical harmony is not a "natural phenomenon," but a highly conditioned and specialized field. It is the material of musical expression, for which we, in our civilization, have an inborn inclination and need. This need is cultivated and furthered by existing trends in our music and musical education.

He could not have been more wrong if he tried. If we express the minor triad and the dominant seventh chords in the tempered scale we get the same insidious complexity we got from the major triad with even more catastrophic results. While **4:5:6** is the major triad he is not quite correct in expressing the minor triad as **5:6:15.** Since he expresses the major triad in close position the minor triad should be as well, thus **10:12:15.** He misses the dominant seventh completely as **7** has never been a part of the harmonic system. It sounds flat because it doesn't belong and is never, ever used and never has been used. The dominant seventh that is used and has been since its appearance is, in root position, **36:45:54:64.**

The eleventh overtone is near to the perfect fourth .

$$^{11}/_{8} = \mathbf{1.375} \text{ is near to } ^{4}/_{3} = \mathbf{1.3333}$$

It is not a member of the enharmonic system but is occasionally used by the trumpet when a true fourth is difficult. Because of this it is often referred to as the 'trumpet fourth'. It is much less disturbing to the ear than **7.**

His use of **16:19:24** makes absolutely no sense. The last thing we would want to do is let the tuning of a chord approach the tempered scale. He is completely in the dark about the leading tone. The leading tone, related to the root of the scale is **15**, *and this is what the violinist plays, always*. **135,** which represents one form of the minor second *above* the root (the object of resolution of the leading tone) is also the alternate value for the minor second that exists to allow a comma shift when moving to anther key. The third of the tempered scale actually replaces the Pythagorean third (**1.265625**) and not the proper third (**1.25**)

$$(\sqrt[12]{2})^4 = 1.2599210498948731647672106072782$$

$$\frac{1.265625}{1.2599210498948731647672106072782} = 1.0045272281986262301163137248 52$$

$$\frac{1.2599210498948731647672106072782}{1.25} = 1.0079368399158985318137684858226$$

It is the tempered scale and its offshoot, the tonometric system that is the pseudo-scientific attempt. Making the chromatic system metric is absurd. Distances, weights and volumes make sense when made metric as they are fixed quantities. A meter is always a meter. A liter is always a liter and a gram is always a gram. They are constant, they don't become smaller as their values increase. The harmonic system, as it exists, is far more than 'simultaneous pitch-assemblages varied in time sequence'.

$^{5}/_{4}$ is a major third. So is $^{75}/_{60}$ or $^{2025}/_{1620}$. While these ratios all produce the same decimal quantity the intervals themselves are significantly different and attempts to make them metric are pointless.

TWELVE TONE MUSICAL COMPOSITION

The dodecaphonic or twelve-tone system of musical composition arose from conditions that actually were not musical and not for the reasons usually put forth. When the Western traditions began in the Middle Ages formal music was largely created in and for the church. There was, as there always is, a traditional folk music and later the music of the Troubadours and Trouvères. As stable monarchies arose we got as well the music of the court. This continued through the eighteenth century. Hayden and Mozart were primarily court musicians. In the nineteenth century things changed considerably as the composer emerged as a popular figure. We had the age of the 'Composer as Hero'. This was the age of the romantic virtuoso and when opera became the spectacle entertainment.

In most societies music and musical traditions are quite static. The East Indian Raga is quite the same today as it was centuries ago. Perhaps because of the invention of musical calligraphy our music went through what appeared to be evolutionary changes. These changes are usually more fashion than actual. There were three distinct periods each of which was considered in its time to be quite unique. Medieval music progressed from the plainsong of the chant to the organum to true linear counterpoint.

Until the early twentieth century musicologists considered the complexities and dissonances of medieval composers to be a crude lack of skill. This is the result of looking at earlier cultures through contemporary eyes. Early composers possessed consummate skill. The age of plainsong was not called the 'Golden Age' for no reason. These composers were quite aware of the vertical sounds of linear counterpoint. The difference between linear counterpoint and sixteenth century counterpoint was the enharmonic system. Where sixteenth century counterpoint was governed by the changing fundamental, linear counterpoint did not change the fundamental. The use of vertical sounds for the medieval composer was a palate of sounds he used for expression, just as with the melodic lines themselves. Such composers were quite aware of the dissonances they produced. The big difference was that the medieval composer did not think harmonically.

In the late fifteenth and early sixteenth centuries the style had become so complex as to be virtually unworkable. Bewildering complexity, just as with many contemporary ideas, were taken to be the mark of artistic creation.

Periods such as this are not uncommon historically. The human need to codify and standardize things arises and imagination, the basic stuff of creativity, is pushed aside and the mark of a skilled practitioner is his spouting of official dogma and memorization of endless and closely held models. Wagner's Die Meistersinger deals with just such an historical situation

This gave way to the harmonic simplicity of the Elizabethan period, which led to the baroque and its changing tonalities, which again became very complex. While not so complex as the late Medieval Period it did lead to the more formal Classical Period. The early classical period was again a simplifying trend.

From the Classic Period came the tools that were to make the nineteenth century Romantic Period so vibrant. Among these were the roots of the modern symphony orchestra and the concert grand piano.

One rather unique feature of Western music is that appears to evolve (it really doesn't but we like to think it does). If such pseudo-evolution occurred in other music of world we cannot tell, as the Western system is the only one we know of with calligraphy so we have a history of its development. All other forms of music on the planet are passed from generation to generation by rote so all we have the most recent style and form of such music. Tribal societies are as a rule quite stable so it may well be that the music of the ¡Kung of the Kalahari is the very music that their ancestors would have made in the stone age.

Another rather unique occurrence in the later nineteenth century was the rise of the scientist as hero. Just about the time the scientist as hero was in full bloom the electronic recording and the motion picture radically changed the landscape of the performing arts. Opera gave way to the movies as the spectacle entertainment and the artist as hero migrated into the popular culture.

The so-called fine arts had entered its 'Meistersinger phase' and moved into the university scene. Since the vibrant thing in university life during this time was science this is the format that 'serious' music adapted. This gave us dodecaphonic pointillism, musical dadaism, musique concréte and their various manifestations as well as many well intended but totally misdirected works such as the Schillinger System. Composition became more like solving math problems than imaginative musical expression. Composition by formula is always the mark of these periods. Composers who chose not to adhere to this risked academic excommunication.

By the middle of the twentieth century the shibboleth of day was that if a composer's music became even the slightest bit popular he was judged to be pandering to the public. The reasoning seemed to be that if calculus is obscure to most people so should music be obscure. This was very convenient for such composers. So long as you played by the rules there was no way you could fail. The price for all of this of course was that there was also no way you could succeed. The musical metric system fit this very well. 'Serious' composers banded together and wrote in obscurity justifying this by saying they were writing for posterity, when people would finally come to understand their music. What they did not realize, as someone so wisely said, is that music that is obscure in one generation is gone in the next.

The most durable composition style of the post-romantic school was the twelve-tone system. Schoenberg devised a system of musical composition based on, as he put it, 'twelve tones related only to one another'. This is a fallacious premise, as the tones of the tempered scale are not intrinsically related in any way (to begin with they do not exist). They are no more related to one anther than would be thirteen isometric tones of the octave, or twenty-seven or, for that matter, any number. There only validity comes from the chromatic scale they were designed to replace. As long as distinct pitches are used atonality is physically impossible. Any combination of frequencies will create one fundamental. Even if the ear cannot discern this fundamental, it is still there.

This really peaked in 1950s and 60s when 'avant-guard' actually become a style of music composition. Fortunately this waned as the century came to a close. Modern trends seem to show a return to real music. In our universities motion picture music, various types of 'popular' music and folk music is now quite acceptable. The thing that holds back a really fine period of composition from developing is the tempered scale and its dreadful offspring, the synthesizer.

The acceptance of irrational values is not confined to music. In most all cases it is the product of linear thinking and such thinking often leads unsatisfactory and unusable results. A good case in point is the Golden Ratio.

THE GOLDEN RATIO

To start with the Golden Ratio is not a ratio. While we do at times use a rational approximation for an irrational number, such as using $\mathbf{22/7}$ for **π,** we never make the mistake of calling **π** a ratio. The Golden Ratio is typically symbolized by the Greek letter **φ** which, as with **π,** gives it a reality that it really does not possess.

In mathematical terms **φ** is expressed thus:

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887498948482045868343656\textbf{4}$$

Two quantities can be said to be in the golden ratio if "the whole is to the larger as the larger is to the smaller". Euclid spoke of it as the Golden Mean and described it thus: "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser".

This can only be accomplished if we divide the line with an irrational number. Written algebraically it looks like this:

$$\frac{a+b}{a} = \frac{a}{b}$$

which becomes

$$\frac{a}{b} = \frac{b}{a-b}$$

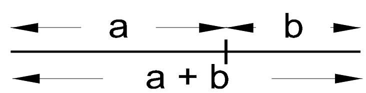
then

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} + 1$$

and finally

$$\frac{a}{b} = \varphi$$

There is no rational solution to this equation. Things become clearer if we look at it geometrically.



a* + *b* is to *a* as *a* is to *b

Simple algebra shows us that **φ** is irrational.

$$\frac{a}{b} = \frac{b}{a-b}$$

If *a*/*b* is a fraction in lowest terms, then *b*/(*a* − *b*) is in even lower terms — a contradiction. The only solution for the equation

$$\frac{x}{1} = \frac{1}{x-1} \text{ is } x = \varphi$$

$$\frac{1}{1.61803398874989484820458683} = 0.61803398874989484820458683$$

We can also observe that $\varphi^2 = \varphi + 1$. From a purely mathematical viewpoint the Golden Ratio is quite intriguing. It is not hard to see why mathematicians find it so compelling. The equation $\varphi = x = 1 + 1/x$ can be expanded recursively to obtain a continued fraction for the Golden Ratio.

$$\varphi = [1; 1, 1, 1, \ldots.] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots\ldots}}}}}$$

This is rather elegant way to describe irrational numbers. The successive convergents of these continued fractions are ratios of Fibonnaci Numbers. This is where things start to get interesting.

THE GOLDEN RATIO AS GEOMETRY

We also see this in the succession of powers of **φ.**

$$\varphi = \varphi$$

$$\varphi^2 = \varphi + 1$$

$$\varphi^3 = 2\varphi + 1$$

$$\varphi^4 = 3\varphi + 2$$

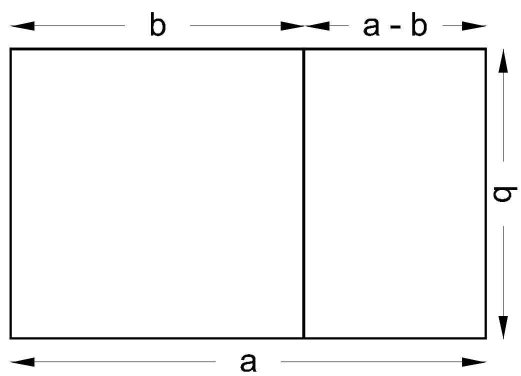
$$\varphi^5 = 5\varphi + 3$$

$$\varphi^6 = 8\varphi + 5$$

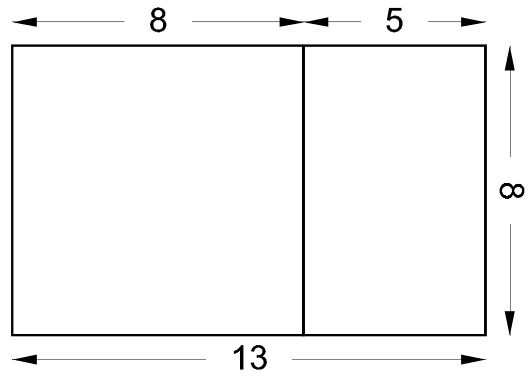
Both integer columns are Fibonnaci Numbers. This leads us to geometry and the relationship of the Fibonnaci Numbers to the Golden Ratio. Johannes Kepler wrote:

> "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."

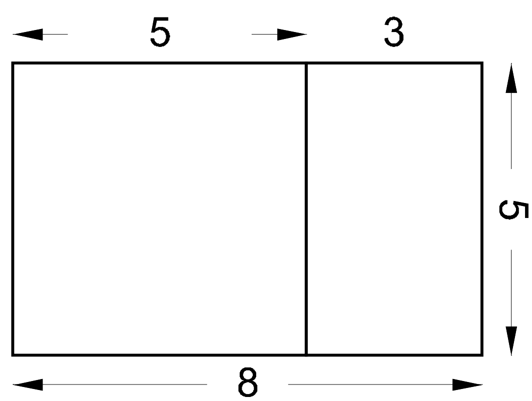
In virtually every case the Pythagorean theorem and in all cases the Golden Ratio is irrational. The Pythagorean theorem when used as the distance formula is hard to replace with anything using rational numbers, or at least nothing as convenient as the distance formula. The Golden Ratio is quite something else.From the Golden Mean we get the Golden Rectangle.



In the drawing **φ ≈ 1.62.** This is an error of about **1.001215**, less than the thickness of the lines of the drawing. If we create a similar graphic using the Fibonnaci pair, **8** and **3** we obtain:

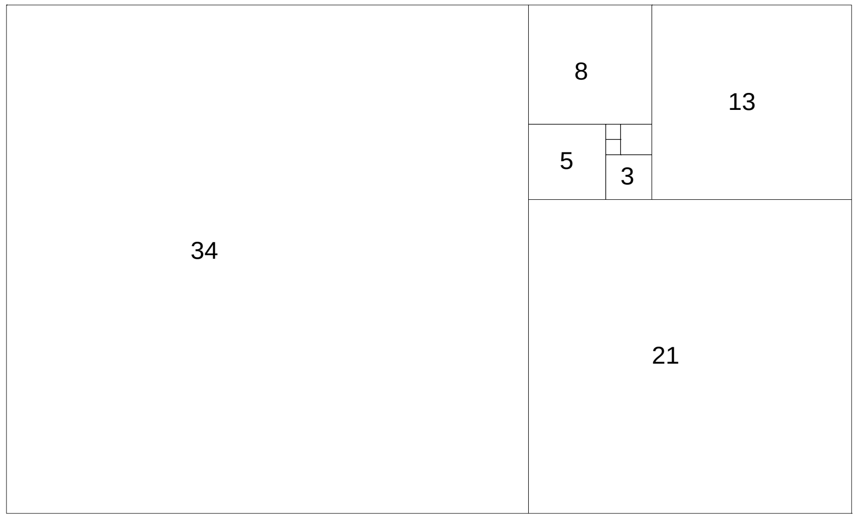


This is virtually indistinguishable from the previous graphic. Even the simpler Fibonnaci pair **5** and **3** creates a graphic in which the difference between it and the Golden Rectangle is virtually indistinguishable.



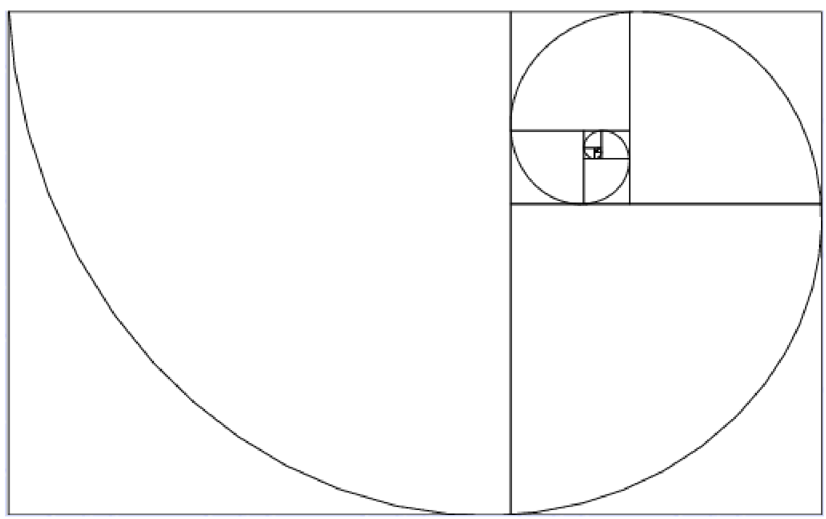
The differences become even more difficult to detect when the graphic representation grows larger as with buildings.

A graphic represent the Fibonnaci series, as a series, is this:



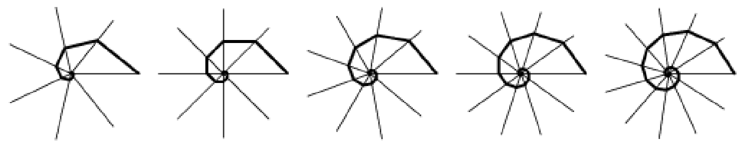
A similar representation of the series derived from **φ** would be actually impossible to draw, however it would look almost exactly as the above image.

This image when based upon **φ** is said to define the logarithmic spiral, often called the golden spiral.



If we were to use the image derived for the Fibonnaci series we would obtain a spiral, which, as with the image itself, would be discrete. A spiral fashioned with either the Fibonnaci rectangles or the Golden Mean rectangles would be visually indistinguishable. An approximate representation of either spiral can be fashioned by connecting the diagonally opposite points of each rectangle with an arc. When this spiral occurs in nature it will be discrete. Logarithms are another of our many abstract tools, useful but not real. As with the rectangles, the spiral drawn from the Fibonnaci series and the one drawn from the Golden Ratio are virtually indistinguishable.

Just a circle can be described as a polygon with an infinite number of sides; the logarithmic spiral can be fashioned in quite the same manner:



A curved line is an abstraction for the same reason the circle is and thus any true geometric spiral is abstract.

THE GOLDEN RATIO IN ARCHITECTURE

While the ancient world had knowledge of trigonometric relations it is sometimes difficult to tell to what degree they used them. For instance there do not seem to be any theodolites much before the sixteenth century even though the ancients had a remarkable understanding of trigonometry. For us to say they used the Golden Ration as a tool is, perhaps, attributing to them a sophistication they did not possess or tools that they did not need.

Perhaps the most prominent example of a structure that was supposed to have been crafted by use of the Golden Ratio is the Parthenon. Here is a picture used to illustrate this.

The white lines are either the representation of **φ** as a series or a graphical representation of the Fibonacci Series. If we think of each rectangle as being proportionally equal we can state that the large rectangle is a Golden Rectangle. On the other hand if we assume this is a Fibonacci series then we say that if we divide the smallest rectangle into two squares the largest square would eight times as large as the smallest and the width of the Parthenon would be thirteen times the smallest square.

Of one thing we can be reasonably sure, neither Ictinus, the chief architect, nor his associate Callicrates, used a design based on **φ** to set the size of the front of the Parthenon. Vitruvius, whose knowledge was drawn heavily from both Greek and Roman architecture, never mentions it in the ten books. If it had been a tool of the ancient architects one would think he would at least referred to it.

If we assume this is a Fibonnaci series then the basic unit of measurement would be the width of the smallest rectangle. This width would equal half the length of the smallest rectangle. This basic square would then provide all the larger measurements. The height of the columns would be **5** units, the height from the top of the columns to the bottom of the roof would be **1** unit and from the top of the columns to the tip of the roof, **3** units. The width would be thirteen units.

In all probability they used neither the Golden Ration nor the Fibonnaci series. My guess they made all the rectangles equal to **8:5 (**or **5:3)**. We shall see why later in the article.

Discounting $\frac{1}{1}$ the first four Fibonnaci pairs are ratios found in music.

$\frac{2}{1}$ =	2.0	Octave	$\frac{5}{3}$ =	1.66666…	Major Sixth
$\frac{3}{2}$ =	1.5	Perfect Fifth	$\frac{8}{5}$ =	1.6	Minor Sixth

This has really no actual connection to music theory, as theory, but architecturally they are significant. The ratios of the various dimension of any structure greatly affect the stationary acoustic structure that forms within them. These basic ratios appear in many areas of the natural world.

There is no evidence that the Golden Ratio was used at all in world before the Renaissance (if indeed it was used then). The most significant architect of the ancient world was the first century A.D. Roman architect, Vitruvius. Vitruvius is quite explicit about the dimensions that buildings and theaters should employ. No mention is made of the Golden Ratio. The numbers he specifies are all the basic small integers that are also the building blocks of the enharmonic system.

The Ten Books on Architecture by Vitruvius were the Bible of architects from the first century, when they were written, until well into the Renaissance. All the great cathedrals were constructed using the Ten Books as a guide. The manuscripts of the books survived due to the Scriptos of Charlemagne who maintained a library for such books.

A recent study of the Cathedral of Notre Dame of Paris uncovered the fact that the ratios between the various dimensions of the cathedral were the simple ratios the Vitruvius set forth in the Ten Books on Architecture. This should not have surprised them as it apparently did.

If we were to construct a rectangle with sides equal to 5 and 3 the ratio produced would be **1.6666….** This is very close to the Golden Ratio. This fact has not gone unnoticed and, in fact, is mentioned often in treatises about the Golden Ratio. Unfortunately we are glued to **φ** just as we are glued to the notion that the concert A must be **440Hz**.

The difference between $\frac{5}{3}$ and **φ** is **1.0300566479164914136743113906094**. This is a very small difference. As a musical interval it lies between the enharmonic comma (**1.024**) and the small minor second (**1.04166666666667**). Audible? – yes. Visual? – I doubt it. In both cases the ear would respond to the difference as would the eye but the brain would most likely ignore it.

Let us assume the dimensions of the front end of the Parthenon are equal to **50** feet and **30** feet. It would look like the Parthenon. The non-Golden Ratio would be **1.6666**… Suppose we want this front end to produce the Golden Ratio. One method of achieving this would be to make the height a hair over ten inches higher. The result would not look conspicuously different from the **50’** x **30'** version. If we were to adjust both the height and width simultaneously the changes would be only a couple of inches.

5:**3** as a ratio was quite well known to the Ancients. This fact is mentioned often by many writers. We can never really know but it would seem much more likely that the designers of the Parthenon used **5:3** or **8:5,** just as did later architects.

A thought about ratios. The fact that $\frac{5}{3}$ = **1.66666…** is, structurally or visually speaking, rather meaningless. What these decimals can provide is shorthand for manipulating these fractions. Apart from this there are no intrinsic properties in these decimals.

Starting with the innermost rectangle the decimal values of the successive ratios ($\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5} \ldots\ldots\ldots \frac{w}{h}$) present an interesting pattern.

(1.0)
2.0
1.5

1.6666…
1.6

1.625
1.615384615384615384615384615384 62

1.619047619047619047619047619047 62
1.617647058823529411764705882352 94

1.618181818181818181818181818181 82
1.617977528089887640449438202247 19

This is an asymptotic series that really begins with **2.0**, as **1.0** merely defines the unit length. It must be remembered that defining the ratios as decimals is the same as defining them as intervals, as would done with sound. **1.625** does not define the rectangle, **13:8** does. The same is true for intervals and frequencies. The first pattern, **2.0** and **1.5**, defines the basic shape of the Fibonnaci pairs, one high then one low. The pattern **1.6666**.. and **1.6**, in a manner of speaking, marks the end of the series, as every successive pair is a small and progressive deviation from the ratios of this second pair.

The top of every pair becomes progressively smaller than **1.6666…** and every bottom member becomes progressively larger than **1.6**. Now mathematics takes over completely and these decimals take on a meaning that they really do not have. The members of the successive pairs are coming, numerically, closer together. These numbers now become a mathematical series that converges to **φ**. The changes on both the top of pair and the bottom are both asymptotic. **φ** is a true abstraction.

In mathematics this is represented by the equation:

$$\sum_{n=1}^{\infty} | F(n)\varphi - F(n+1) | = \varphi$$

For instance, a function that expands to **φ** in no way makes **φ** real. The same is true with ∞. In truth, $\frac{1}{\varphi}$ is just as physically impossible as $\frac{1}{\infty}$ and for the same reasons. The problem with choosing **φ** as the Golden Ratio arises because we see the decimals as objects instead of markers for the height and width of successive rectangles. This is the way we do things mathematically and in this case we are looking through the wrong end of the telescope. We are looking at this convergence absolutely backwards.

We can use these decimals to define the amount of change between each progressive pair, for instance, the 'distance' between **1.625** and **1.6153846153846153846153846153846 2**, which is

$$\frac{\mathbf{1.625}}{\mathbf{1.6153846153846153846153846153846 2}} = \mathbf{1.005952380952380952380952380952 3}$$

If we chart these we get

$\frac{5}{3}$ (**1.66666666666666666666666666666667**)

<u>1.04166666666666666666666666666667</u>

$\frac{8}{5}$ (**1.6**)

<u>0.98461538461538461538461538461538 5   (1.015625)</u>

$\frac{13}{8}$ (**1.625**)

<u>1.00595238095238095238095238095238</u>

$\frac{21}{13}$ (**1.61538461538461538461538461538 46**)

<u>0.997737556561085972850678   (1.002267573696145124716553287 98)</u>

$\frac{34}{21}$ (**1.61904761904761904761904761904 76**)

<u>1.00086580086580086580086580086 58</u>

$\frac{55}{34}$ (**1.61764705882352941176470588235 29**)

<u>0.99966953073364177131526768010   (1.000661266306946246841 06)</u>

$\frac{89}{55}$ (**1.61818181818181818181818181818 18**)

<u>1.00012626262626262626262626262626</u>

$\frac{144}{89}$ (**1.61797752808988764044943820224 72**)

This series converges to **1**. This makes sense, as the Fibonnaci Series is a set of ratios whose numerator and denominators are increasing in value. As the fractions progress they approach x/x. As they increase, the ratio between them converges to **1.0**. We can now interpret the Golden Section and the Golden Ratio from a different viewpoint. The convergence of successive ratios to **φ** describes successive moves away from something not toward it. The truly remarkable thing about the series is that the every successive pair is a slight deviation from the pair, **3:5** and **5:8**.

This deviation is always so small that the eye would see every Fibonnaci rectangle as looking like **3:5** or **5:8**. This is the magic pair. We have a Greater Golden Ration (**3:5**) and Lesser Golden Ratio (**5:8**). The Golden Ratio is either **1.666666…** or **1.6**, not an abstract point between them. **3:5** would be the preferred version simply because it is made up of smaller numbers. Ockham's Razor**!**

The mere fact that we are dealing with the numbers **3** and **5** causes us to take a look at music and its effect on architecture.

Structurally the Fibonnaci Series stops at **8**. Consider our **50' x 30'** Parthenon. It is now **8:5**. If we altered the dimensions to **49' x 31'** we would get a ratio very close to **5:8.** It would take keen eye to distinguish between **5:3** (**50'** x **30'**) and **8:5** (**49'** x **31'**). Of all the consecutive pairs of ratios in the series this has the most detectable difference. Each successive change is less than it predecessor. Starting with **5:3** every Fibonnaci rectangle looks to the eye as either **5:3** or **8:5**. Some similarity between graphics and music can be seen when we express both as decimals. **13** does not enter the graphic system as it produces **1.625** and both **5:3** and **8:5** produce more useful ratios. The same is true in music for the same mathematical reasons. In a scale with the fundamental = **1**, **13** produces the note **1.625**. The system already has the more useful

$$5/3 = \mathbf{1.66666...}$$

or

$$8/5 = \mathbf{1.6}\ .$$

In music the ear simply will not tolerate anything above **8** in Fibonnaci Series. The entrance of **13** into the enharmonic system creates dreadful beats, which, in turn, spawn even more dreadful second-generation differential tones. The higher frequencies become even worse. On the other hand it would seem that the eye is not so meticulous. I suspect that the eye accepts everything from **5:3** to that nebulous fraction that produces **φ** and probably interprets it the same, quite likely as **5:3**. Indeed, would **5:3** really be more pleasing to the eye than **89:55**? Is such a ratio even part of the brain's interpretation? I am guessing that it is and the brain prefers **5:3**.

Another interesting convergence arises if we explore the ratios formed between **3:5** and each successive Fibonnaci pair.

$$\frac{\frac{5}{3}}{\frac{13}{8}} \quad \mathbf{1.025641025641025641025641025641025641 03}$$

$$\frac{\frac{5}{3}}{\frac{21}{13}} \quad \mathbf{1.031746031746031746031746031746031746 03}$$

$$\frac{\frac{5}{3}}{\frac{34}{21}} \quad \mathbf{1.029411764705882352941176470588 24}$$

$\dfrac{\frac{5}{3}}{\frac{55}{34}}$	**1.03030303030303030303030303030303**
$\dfrac{\frac{5}{3}}{\frac{89}{55}}$	**1.02996254681647940074906367041199**

This converges to **1.0300566479164914136743113906094** which is the difference between **5:3** and **φ**. This gives a better illustration of the nature of this convergence.

MOTION, ENERGY, GRAVITY and TIME

It should not be surprising that a treatise on the enharmonic system should progress to motion, energy, gravity and time, as music exists as a result of all of them. Even with all of the wonders of today's science there are four fundamental things we cannot really explain, motion, energy, time and gravity. The basic components of music, which is vibration, are motion, energy and time. Gravity, of course, affects everything and is perhaps the most mysterious thing of all.

The most difficult thing to explain about motion is inertia. Why does an object once set in motion continue to move until another force changes that motion? We cannot sense motion. We can only sense changes in motion. We say that the people in an orbiter are traveling at seventeen thousand miles an hour but those in the orbiter feel absolute no sense of this motion. If they had no instruments and could not see out they would have no way of knowing if they were moving or not. On the other hand someone on the earth definitely sees them moving at that colossal speed. Since neither the occupants of the orbiter nor the observer on the ground have any sense of motion something in the orbiter must have changed when it was accelerated to that speed.

In the Doxographists, Plato referred to a Pythagorean concept of motion:

Motion is a certain otherness or difference in matter. [This is the common limit of all motion.]

This may be as good a definition as we will ever have. When we consider motion in terms of what it does, that is, change position in space, we run into the paradoxes, the most famous of which are those of Zeno of Elea some twenty-six centuries ago. These are put forth as paradoxes of motion when in fact they deal with continuity and thus are applicable to any form of continuous structure - such as time or fields.

In order to cross a room I must first travel half the distance, then half the distance again and then again and on and on. By definition I cannot cross a room. This quickly becomes simply another argument as to the nature of the infinite series, in this case:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\ldots\ldots\frac{1}{n}$$

If we are going to use mathematics to describe our projections of physical reality we must observe the rules. It is mathematically impossible for me to cross the room - except for one thing. This is better observed in another version of the same paradox.

A tortoise and a hare move toward a line. The hare begins twice as far as the tortoise from the line and moves twice a fast ergo, the hare will cover twice as much distance as the tortoise in the same interval of time. The first move is half the distance. This distance can only be specified if they both *stop.* As long as an object is moving it cannot be said to be at any given point at any given instant in time. Such a thing as an instant is an abstraction just as is a point. We cannot stop time as we can motion so an instant cannot be specified because of the seeming continuous nature of time. It is always the next instant. There exists no point that can be said to be half the distance, as time does not stop on points.

As long as neither the hare nor the tortoise stop the hare has only to travel at a speed that is greater than that of the tortoise. The distance each is from the end point is continuously changing so the distances that mark the infinite series cannot be specified. Such fractions as $\mathbf{1/2}$ or $\mathbf{1/4}$ of the distance occur only if they stop. No matter how far behind he is or how slight his speed advantage, the hare will always catch and overtake the tortoise.

Let's look at motion from a different perspective. In order for an object to be said to have moved it must be in a different position from what it was before it moved. If we let **A** be in position **1** then motion can only have occurred if position $\mathbf{B} \neq \mathbf{A}$. As such there will always be a distance between position **A** and position **B**.

Regardless of the size of this distance, since it is both finite and real, it can be divided by **2**, and then by **2** again and on and on. These subdivisions cannot become infinite because then the object would not have moved. It is impossible to move in infinitely small steps. There are well-formed arguments that state that motion cannot be a continuous function. Exactly the same argument can made in respect to time and the same conclusions be drawn.

The first grand oxymoron in the incredible system that is built on oxymorons, namely calculus, is instantaneous speed. On the other hand, it may well be the most useful fiction in all of mathematics. There are no instances in time just as there are no points in a distance. While time and motion are intimately entwined there is a difference between them. Motion can be stopped anywhere. The only way to 'stop' time is to move at the speed of light. Even if we could isolate an instance the distance would be **0** and thus, so would the speed. So we let the clock run a tiny bit so now our instance has length and so reality. It's length is $x+\Delta x$, where Δx is, re Albert Einstein, a very small number, arbitrarily small says the mathematician. Make them small enough and we have value that is very 'close' to what the 'speed' would be at x if, indeed, instantaneous speed actually existed. Relativity tells us we can never accelerate to the speed of light. The closer we get the light speed the greater our mass and the slower our clocks run. The quanta of electromagnetic radiation do not accelerate; they are created already traveling at light speed.

If a body is moving at a uniform speed the notion of instantaneous speed is unnecessary. A falling body is continuously accelerating and it is natural to wonder how fast it is going and any given point. What we are really suggesting that if the acceleration could be stopped at a given point and the motion of the falling body made constant how fast would it be going. Instantaneous speed quickly became 'the derivative of a real value function' and we were off to the races.

The common factor in time, motion, energy and gravity is that we can only define what they do, not what they are. Another thing quite likely common to all of these and one that our linear mathematics does not show is structure.

STRUCTURE

There are a number of examples that would seem to indicate structure. Consider a tuning fork. If one tine is struck the opposite tine will move instantly in opposition to the struck tine. The question is why does the second tine move at all? The usual explanation is that the energy travels from the struck tine to the second tine and causes the second tine to move. This says absolutely nothing. What does it mean 'the energy travels to the other tine'? And why does it 'make ' the other tine move?

We usually get the same explanation for the movement of the balls in a device called Newton's Pendulum.

The common simple explanation for the behavior of Newton's Pendulum is that the 'energy' travels from the first ball through the middle balls resulting in a movement of the last ball. Formulas using the conservation of momentum and the conservation of kinetic energy tell us that velocity of any moving sphere after the collision is the same as before the collision. It still does not tell us what is actually happening. What does it mean 'the energy travels though the line of balls'?

The classic definition of energy is "*energy is the ability to do work*". As is so often the case, that definition tells us what energy does, not what it is. We do not need to know what energy is in order to work with it. Consider kinetic energy. This is the energy a body contains as a result of its motion. It can be expressed with a very simple formula.

$$\mathbf{E_k = \frac{1}{2} \mathit{mv}^2}$$

Where $\mathbf{E_k}$ is the kinetic energy, *m* is the mass of the object as *v* is its speed. This is a very practical and useful formula; however, let's look at it from a different perspective. It requires a vast prolonged force to put the shuttle into orbit. The crew senses this throughout the entire burn of the rocket. Then the rockets are turned off. The shuttle now, from the shuttle's point of view, is at rest. The crew of the shuttle has absolutely no sense of motion and yet, from the viewpoint of those still on the earth the shuttle is moving latterly at about seventeen thousand mile per hour and is falling at **32** feet per second per second. (**32'x** s^2)

When the retrorockets fire the crew, who are at rest, sense this as acceleration in a direction opposite from the initial acceleration. They certainly feel that they are slowing down.

We state that the initial force was converted into kinetic energy but if the crew were completely isolated from any outside influence they could find absolutely no indication of this kinetic energy. On the other hand something has changed in the shuttle. It sees the earth moving very quickly beneath them. Its clocks are running at a different speed than those on the earth.

We state that not only does the shuttle have kinetic energy but also potential energy. This view is only valid when there is a difference in the motion between the earth and the shuttle. If everything else in the universe were to vanish conditions in the shuttle would not change. They would consider themselves at rest.

What is interesting here is that they would sense the exact same conditions in a gravitational field as without one. We will return to gravity later. Consider Einstein's famous equation $\mathbf{E} = \boldsymbol{mc}^2$. **E** of course is the energy. It is measured in joules, which is the SI unit of electrical, mechanical and thermal energy. It is usually defined as a unit of electrical energy equal to the work done when a current of **1** ampere is passed through a resistance of **1** ohm for **1** second or a unit of energy equal to the work done when a force of **1** newton acts through a distance of **1** meter. This covers all bases so far as the science goes. ***m*** is the mass measured in grams and ***c*** is the speed of light in a vacuum, measured in kilometers.

The formula is usually understood to mean that mass can be converted into energy. Mass is a collection of atoms but once again we are faced with energy and what it might be. It is stated that an atomic bomb releases vast amounts of energy. Just as with the energy traveling through the tines of the tuning fork or through the successive spheres of Newton' Pendulum it says virtually nothing.

When the bomb explodes there is a shockwave created by the rapid expansion of the mass of the core. This compresses the surround air and a traveling wave is formed. This happens in any explosion. Bombs are also said to release heat. Heat is a condition. It is the result of atoms moving (vibration) faster. What must be released is something that causes the motion of the atoms to increase. That something is electromagnetic radiation. As many who have been near to explosions will testify you can be badly burned even when you are beyond the range of the effective shockwave and fragmentation.

The bulk of the 'energy' that is released from a fission bomb is in the form of soft X-rays. The air surrounding the bombs is opaque to these rays and when it absorbs them the air becomes extremely hot and reradiates soft X-rays. This continues and reoccurs in very brief time segments. It is known as radiation transport.

The sun does not release 'heat'. It too releases electromagnetic radiation. We know that electromagnetic radiation is discrete. Plank describes the 'heat quantum'. What he really is describing is the discrete structures of radiation.

Atoms are structures composed of other structures. When fission occurs nothing is released, it is restructured. Even in string theory the strings are discrete vibrations. It appears safe to say that we never encounter so-called *pure energy*. Quite likely there is no such thing, or at least nothing that we can define. We are dealing with structure. The very term energy may be superfluous.

We often speak of 'pure energy'. We never encounter energy that is not structured. What it is that these structures consist of is still an unknown. What we call energy is all that really exists in physical reality. If we break it all down far enough there is noting that exists but energy and we have no idea what that energy is. We may never know. As long as we understand that it is structured and what these structures this lack of knowledge becomes philosophical.

Back to the tuning fork and Newton's Pendulum. The tines of the tuning fork are symmetrical and are of uniform density. The spheres of the pendulum are all the same size and density. When one tine is struck the other tine moves in the opposite direction with the exact period as the struck tine. Due to the shape of fork it cannot be any warping of the fork that causes this. A similar thing occurs in a bell where the side opposite the point of impact moves opposite the point of impact. The other sides move out when the initial sides move in. The sides form a ring and the moving is easier to explain.

With the fork however this is not the case. Contrary to what is often stated the motion of the tine is not simple harmonic motion. This would only be true if the tine did not bend. When the tine bends it is deformed and very powerful internal structure is formed within the tine. Although it would be slight in all probability the tine itself undulates. This structure is longitudinal in nature. We call it a wave but that is largely because that is the way we measure it. If we use analog devices to make the measurements we will get analog results.

We can say that it is an energy structure although that still leaves us with no idea of what energy is. The logical explanation of the movement of the opposing tine is that this longitudinal structure travels down the stem of the fork. A conjugate is returned which, being a conjugate enters the opposing tine and, again by the fact that it is a conjugate creates a mirror of the initial structure. It is the structure then that causes the second tine to vibrate.

The bending of the second tine also creates a similar wave that travels back down the stem and another conjugate travels to the first tine. This feedback loop is why the fork vibrates for such a long time. There would be a very tiny but cumulative delay between these exchanges. It may be that this gradual going out of phase does more to stop the vibration that the friction that is usually stated as the cause of the decay. This then is a clear example of motion being caused by structure.

We could construct a version of the pendulum using just two spheres. In this case the moving sphere would strike the object sphere and the object sphere would swing out the other side just as is does with several spheres. The significant thing is that the moving sphere does not decelerate and the object sphere does not accelerate. The object sphere continues the motion of the moving sphere just as if the initial sphere were swinging alone.

The same thing occurs in line of spheres. The last sphere continues the motion of the initial sphere just as it does when the moving sphere strikes a single sphere. If two spheres are drawn back and released two spheres leave the other side with the same momentum and speed as the two initial spheres.

Rather than a vague explanation that it is the energy traveling through the respective spheres it is more logical to assume that, as with the tines of the tuning fork we are dealing again with structure. Rather than calling it a structure of energy we shall refer to it as dynamic structure.

If a billiard ball rolling with no spin strikes a stationary billiard ball the moving ball stops instantly and the target ball continues the motion of the first ball. What must happen here is an exchange of structures. Inertia and momentum are the same thing. The same phenomenon that keeps a moving object moving also keeps a stationary object stationary and that something can only be structure.

The fact that there is no acceleration or deceleration in the two billiard balls when they collide reinforces this notion of an exchange of structures. Motion itself must be structural. The otherness in matter that Plato referred to must be structure. The structure that moving the opposing tine in the tuning fork must be definable. If we could define that structure we could make the tine move by feeding the structure to the tine. Since this structure is mechanical it must consist of a frequency pattern, which would be basically the same as those that define the enharmonic system. It is vibration that creates the patterns that create a sound, thus it must have the same structural pattern.

If that is true then the highly efficient exchange of structures in the billiard ball collision must be the same. The same patterns must also create the sound of the collision. If we knew what this pattern was it is quite likely that we would have but feed the pattern to a billiard ball and it would move just as of it were stuck by another ball. This could have profound implications.

The action of Newton's Pendulum is very efficient as the spheres swing back and forth for an amazingly long time. It would appear that it is the friction of the air and the friction of string holding the spheres as it move against the device holding it that is most responsible for the deterioration of the movement. The exchange of structures seems to add little to this decay.

Most of the mechanical machines we use are very inefficient. The majority of such devices are activated by hitting them with something else. An internal combustion engine is operated by hitting the top of the piston with an expanding gas. Guns work the same way. Space vehicles are put into orbit by hitting them on the bottom, again with a rapidly expanding gas.

The collision of the billiard balls and the spheres of Newton's Pendulum show us that this efficiency must come from both the moving and the target spheres being identical in materials (which insures similar densities) and shape. In machines that use expanding gas the force hitting them is amorphous. The gas attempts to expand in all directions and much of the potential force is wasted as heat as well as undirected motion.

It would seem logical in designing mechanical device to insure that the various parts have masses that, with other masses of the device, create ratios that are musical in nature. This alone would increase efficiency. It would reduce unwanted oscillation and would require less energy to maintain the action of the machine.

A classic example of this is the cathedral bell. It is critical that the clapper and the bell proper be masses that form a simple musical ratio between them. We usually state that it is the clapper that feeds an impulse structure into the bell and the sound is formed. It the bell that is moving so it is much more logical to assume that there is an exchange of structures rather than the clapper feeding a more or less undefined impulse to the bell. This is why the ratio of the masses is so critical.

GRAVITY

One of the main problems is science, and to a degree in all fields, is that too many people wish to *be* scientists and too few wish to *do* science. Perhaps it is inevitable but to a great degree our universities are becoming more like simple trade schools. All too often students simply absorb the material they are given, as to do otherwise might displease the instructor and run the risk of a low grade and low grades make it difficult to get a position once college is finished. They often become instructors themselves and regurgitate the material drilled into them and ideas that should be questioned became dogma. True learning can only happen if the student questions every word the professor utters.

The events that occurred at the end of the nineteenth century and the beginning of the twentieth proved that, contrary to long held (and often still held) belief that physical world is not an analog world. Einstein over his life often referred to the fact that the mathematics we use and the actualities of the physical world are quite different.

An interesting example is gravity. One of the natural occurrences that moved Einstein was gravity. He correctly observed that gravity is not a force. While this is obvious it is quite often ignored. A recent documentary on the aspects of gravity had, as its spokesman, a recognized physicist from a prominent university. He began by saying that gravity does not pull you down into your chair, which is true. However he then went on to state, because of the curvature of space around a massive object, gravity *pushes* you into your chair. This statement was from an established physicist. *Gravity is not a force; it is a condition*. It can neither pull you nor push you.

The example Einstein used was a man jumping from a building. Such a man feels nothing pulling him down. If it weren't for the air rushing past him and what he could see as he fell he would have no way of knowing he was actually falling. For those few seconds of the fall he feels exactly what the astronauts feel when in orbit.

When we fall we feel as if we are weightless and yet nothing about our physical makeup has changed. We haven't 'gained' or 'lost' any energy. We speak of conditions in an orbiter as being 'zero gravity' when in fact the orbiter and its contents are just as much effected by gravity, as they would be if they were 'standing still'. Standing still that is, as its motion relates to the earth.

The motion of the orbiter that is at roughly right angles to that caused by gravity does not cancel out the effect of gravity. It merely adds another motion. Disregarding rotation and other movements the orbiter has two primary motions. It is accelerating toward the center of the earth at the rate of acceleration that it would have if it were not moving laterally as to the surface of the earth, and it is moving at a tangent to a point on the surface of the earth directly beneath it. If it weren't falling it would continue on the tangent in a straight line. Since it falling as well, this line of tangent progresses with the orbiter. It is important to be aware that one motion does not 'cancel' the other. These are two separate and distinct motions. As a pair they control the actual change in position of the orbiter.

In spite of the more practical explanation by Einstein that the gravitation 'field' of a body warps or bends the space around we still often hold fast to the idea that gravity somehow pulls things down. That is certainly what the physicist who said the gravity pushes us into the chair was thinking. We still think of gravity as a force created by a given mass. Mass does not create gravity. Gravity is a basic part of every atom in the universe. The atom does not create it any more than a proton or the atom creates a neutron or an electron. You can't have an atom without gravity and you cannot remove gravity from the atom.

If we try to think of gravity as a propagating force we run into conceptual difficulties that defy solution. Newton was bothered by the action at a distance that he assumed gravity seemed to entail. This is only a problem if we think of gravity as a force, which it is not. Gravity, this warping of space, comes into existence when the atom is created. An atom and its gravity are inseparable. Gravity does not propagate. The extent of the gravitational field (we will call it a field event though the continuous field may be a fiction) of an atom is there when the atom is created, it does not need to propagate as it is already there.

Every atom in existence has its own gravitational field. The earth has a large gravitational filed simply because it is made up of a lot of atoms.

Because of the physical nature of an atom it would seem logical to assume that the gravitational field of the atom is structured. The distinct lines that form in solar prominences show structure. If this were to be so then it follows that this gravitational field must also 'warp' anything that is within it. It alters this 'otherness in matter" that Plato spoke of. An object falls in a gravitational field for the same reason that the opposing tine of a tuning fork moves in response to the initial tine being struck. They both move because of a change in the structure that is the object.

Since there is no such thing as a 'fixed point' inertia and momentum are both permutations of the same thing. When we sit in a chair we feel our own weight because we are still falling. The chair does not stop the falling but inhibits the change of position, which is something else again. The fact that we do not change physical position does not mean we have stopped falling.

Consider a billiard ball being pushed by two cues. The points of the cues are on points directly opposite from each other. If the force from both cues is equal the ball will not change position. The common explanation of this is that the forces cancel each other. This is not quite correct. If we increase the force on one of the cues the will begin to *accelerate* in the direction of the stronger force. We will assume the balls and the cues are in space so that we can disregard any friction.

If we then make the forces equal again the acceleration stops but the ball and the cues are moving in relation to the location where they were before we increased the force on the one cue. From the viewpoint of the ball conditions are exactly as they were before this motion was initialed. If it were not for the reference point of the initial position we would have no way of knowing that ball was 'moving' Motion is always relative and always requires a reference point.

If we move one of the cues out of line with other the ball will accelerate in a direction different from the direction of either cue but caused by the combined action of the force of the cues. In physics the directions of the forces are known as vectors and the combined vectors of the cues creates a new vector, the result of the addition of the vectors of the two cues.

The ball is actually moving in two different directions on any resultant vector. It is moving in a straight line in the direction of one force and in a straight line in the direction of the other force. The result of the two simultaneous motions is the resultant vector. An object can have any number of different motions, each independent of one another. The moon has more than fifty separate motions. If we add a third cue on a different spot we have the ball moving in three directions. The actual change of location would be a vector comprised of the movement of the three cues.

The resultant vector will divide the angle formed by the force of the two cues. The acceleration of the cues does not change with the change in the position on the ball. The second law of motion states that acceleration is equal to force over mass, $a = F/m$. Changing the position of the cues does not alter this in any way. As the angle between the cues approaches $180°$ the apparent speed of the resultant vector slows until at $180°$ all perceived motion stops, that is, there is no change of position of the ball. Both balls still continue to accelerate at the same rate. The ball is actually moving in two directions simultaneously. This has nothing to do with the change in position of the ball.

The same is true with gravity. Anything within a gravitational field accelerates. There is no known way to stop this acceleration. As we sit in the chair we feel our own weight, which is a product of this acceleration. Not only is the acceleration caused by gravity not a force but also the action of the chair does not produce force. The acceleration of our body is structurally transmitted to the chair and the chair continues the acceleration just as the spheres of Newton's pendulum. The only force in the action of Newton's Pendulum occurs when the initial sphere is pulled back for the line of spheres. Once the ball is released there is no more force, only momentum.

The chair, in a manner of speaking, gains the weight of our body and this weight too is the product of acceleration. The structure that is the acceleration is transferred to the floor and so on. The acceleration of gravity weakens with the inverse square of distance. It would seem that the gravitational field of an object never ends. If the universe only contained two objects it would seen that they would eventually come together regardless of how far apart they were originally. Thus there is no location in the universe where there is no gravity. All of space is warped to one degree or another and so all matter in space is accelerating to a plethora of different vectors and rates of acceleration.

There is no action at a distance as the warping we call gravity is always there. It is often stated that if the sun were to disappear we would not know it for about eight minutes. This is incorrect. It assumes that gravity is a force and as such must propagate. Gravity is not a force it is a condition, it does not *happen*, it *is*. If we could make the sun disappear instantly the planets would instantly quit falling toward the sun and fly off into space on tangents to the points where they were when the sun disappeared.

Einstein spoke of geodesics when he described this warping of space. Without going into the very complex mathematic of geodesics we can make a casual observation that all free motion in space, that is relative motion with no force involved, is rectilinear. An object is always moving on a straight line, it is the space that is curved. This is, for the most part, what Einstein said and it makes perfect sense.

MOTION AS STRUCTURE

Returning to Newton's Pendulum once again we will take up the mechanism of what is really happening in the function of the device. The common explanation is that the 'energy' travels from the ball doing the striking through the rest of the balls causing the end ball to fly out from the group. The only problem with the explanation is that it doesn't say anything. What does it mean, the energy travels though the line of balls?

If we bring in the conservation of momentum it sounds better but it still doesn't really say anything. Unless we can define momentum we have no viable explanation.

When we pull the initial ball back we are apply a force. That is the only force that is applied. Once the ball is released there is no more force involved. When they are hanging motionless all the balls are accelerating downward. That is what keeps the lines holding them tight and slightly stretched. The rack holding the balls is also accelerating downward as is the table the device is resting on, etc.

The ball, when pulled back is still accelerating straight down. When we release the ball it falls straight down. The line restraining it causes a secondary motion not unlike the dual motions of the orbiter. Thus the moving ball falls into the first object ball. Almost always it will be said that it is the force of gravity that starts the action of Newton's Pendulum but gravity is not a force. Beyond the force that initially pulls the fist ball back there is no force whatsoever in the action of the device.

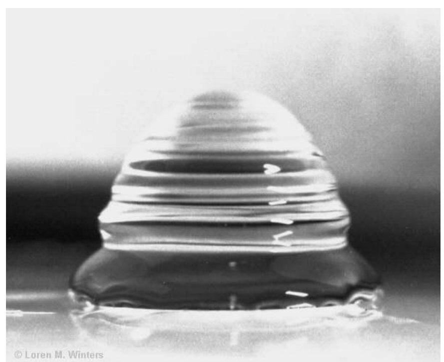
If the ball were allowed to fall straight down there would be no change in the falling ball from when it was tethered but when the string is holding it a new motion is induced, that being a rectilinear motion at right angles to the motion of the fall. In order to work properly all the balls of the pendulum must be identical in shape size and mass. This suggests structure.

Consider two of the balls in space with nothing near enough to be a reference point. Let the two balls be approaching each other with their centers of gravity in line with each other just as when they are tethered to rack of the pendulum. When they collide they would appear to move away from each other at virtually the same speed at which they approached. Since there is available no fixed point we have no way of knowing which ball is moving. In fact such a distinction is a moot point. If we consider one ball at rest we could say that the moving ball stops and the ball at rest begins to move at the same rate of speed and the first ball. The effect I the same if chose the other ball to be at rest. If we consider them to be moving at equal speeds they recoil as was previously mentioned.

The same is true with the balls of Newton's Pendulum. Our assumption that the balls hanging are motionless is artificial. What must be happening is that the two balls exchange structures. The first ball struck exchanges structures the next ball in line and so one until the last two balls exchange structures. With no ball to exchange with the end continues the movement of the initial ball. At this stage we do not know just what those structures are but it gives something a bit more tangible to seek than the rather amorphous energy.

If the were only ball in the universe we could add force and accelerate it. When we remove the force the ball would be exactly as it was before we applied the force. If we say we have added energy to the ball where is it? There would be no way of knowing that a force had been applied and yet something had to change, that certain 'otherness' in matter.

As to the structures themselves I think that the structure that is affected by force is already present in any object. Consider this high-speed photo from Part II.



This clearly shows structure and a rather complex structure at that. As mentioned before the structure was most likely already present to one degree or another in the drop of water. The best candidate for what created this inherent structure would be surface tension. Keeping a drop of water spherical should require force and force presupposes energy.

This presents an interesting question. Surface tension is the result of a restructuring of the electrons in the outer ring that have no other water molecules with which to share electrons. Surface tension then is electromagnetic force.

According to the definition of force that we use this force should require energy but atoms do not seem to ever 'run out' of energy. Much of the water on the planet was here before life itself appeared and surface tension was present from the beginning. To be sure the water is always changing position and, due to the nature of water different molecules are always at the surface but four and a half billion years is a long time.

Then there is the surface tension of solid. When water freezes the conditions that create surface do not change. Also density is a factor in the strength of surface tension. The surface tension of mercury is about nineteen times that of water. This surface tension is strong enough to cause mercury to form spherical drops. The steel spheres of Newton's Pendulum must have a strong surface tension and while it is not strong enough to deform the sphere it would certainly be strong enough to form an internal structure.

It follows then that this is the structure that is affected when the spheres of the pendulum collide. This would explain why a resonance forms so easily in very large structures, such as the moon. When a force is applied to an object it does not have to create the structure as the structure is already there.

This supports the argument that it is a structure formed in the initial tine that moves the second tine. The two tines have structures that are identical, the only difference is that they are mirror images of each other. When the first tine is struck the structure from the impact (often referred to as the impulse function) changes the existing structure of the tine. The structure of the tine is also fed to the object doing the striking. It is this change in structure in the first that is transmitted to the second tine.

Another common misconception about energy concerns a string under tension, or in fact, tension in general. Consider a grand piano. When a tuner tunes a string he applies force. This force stretches the string, which deforms it. The natural tendency of the string is to return to its original shape. Raleigh spoke of this as a restoring force. It is a force and so the string is constantly ‘releasing energy’. Some will argue this and insist that the string only releases energy when it is broken. This is wrong. When broken the structure of the atoms of the string are allowed to return to their normal structure instantly. The combined pull of the strings in a tuned grand piano is more than 35 tons. This certainly means energy. It becomes clearer why the term ‘releases energy’ is not a particularly good expression. If the iron frame of the piano were not strong enough this combined force would tear it to pieces.

Much of the energy that goes into the sound of a piano comes from this stored energy of the strings. When a key hits the string the energy that is being used to attempt to return to the initial structure of the string amplifies the structure that the hammer has transferred to the string. Just as with drums and bells the shape and mass of the hammers are consistent in every piano. If the piano is left untuned, the string will slowly regain its normal position and stop ‘releasing energy’. This takes a remarkably long time. This is a good example of natural amplification and a very effective way to store energy. Because we do think in terms of structure we have never used this very effective energy storage system in anything but springs and stretched strings.

When we apply a force to an object the objects accelerates. Something in the object has changed as it is now moving (again relative to a initial position). Relativity tells us the accelerating object gains mass and it clocks run slower. This is only from the point of view of the observer who does not accelerate. To the object that was accelerated everything looks quite the same.

It is often stated that these ‘relativistic’ effects are only significant when we approach the speed of light. We forget that these effects are not dependent on such great speed. If a person gets up and walks across the room his clock is running slower than the person who remains seated. When we jump into the air the earth itself in effect moves proportionately in the other direction. While these changes and motions are too small to measure and are quite logically disregarded they are still real. The second law of motion is true regardless of the size of the objects involved. If time dilation occurs at speeds near the speed of light it must occur at the speed of a person walking.

The enharmonic system and its mathematics shows us that music, sound and vibration are both discrete and digital. We can surmise virtually everything the physical universe is. If gravity is structure as I suggest it is then the entire universe is actually one huge structure.

This gravitational structure affects everything that is within it or perhaps to state it better everything is part of this one huge structure. Motion then is a change in the structure of the moving object in relation to the structure in which the object resides.

There are many things we cannot do with discrete mathematics, or at least not do efficiently, but there are many vital things that we can do. Contrary to what it looks like we do not live in an analog world; or universe for that matter.

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